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Papers upon analytic sets

A CONVERSE OF THE JORDAN-BROUWER SEPARATION THEOREM IN THREE DIMENSIONS

BY

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Reprinted from the
TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY
Vol. 32, No. 4, pp. 632-657

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That the closed $(n-1)$-manifold immersed in euclidean space, E_n, of n dimensions $(n>2)$, separates E_n into just two domains of which it is the common boundary, was shown by Brouwer in 1912.† That the points of the manifold are accessible from each of its complementary domains Brouwer proved in an accompanying paper,‡ and in the latter connection he gave an example to show that a bounded and closed point set which separates E_n into just two domains and every point of which is accessible from each of these domains, is not necessarily homeomorphic with a closed manifold.

The above results of Brouwer, in so far as the connectivity of the set residual to the manifold in E_n is concerned, received considerable extension at the hands of J. W. Alexander, who not only demonstrated, using modulo 2 Betti numbers, that the residual set is just two connected domains, but that a certain duality exists between the connectivity numbers of the set and those of its complement.§ If we denote the ith connectivity number of a set F by $R^i(F)$, then, for the particular case where C^i is a set in E_n homeomorphic with an i-sphere, Alexander showed that

$$R^i(C^i) = R^{n-i-1}(E_n - C^i) = 2, \tag{1}$$

$$R^s(C^i) = R^{n-s-1}(E_n - C^i) = 1 \qquad (s \neq i). \tag{2}$$

In the case where $n=3$ and $i=2$, relation (1) states that the complement of a simple closed surface in E_3 is just two domains (as also shown by Brouwer),

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* Presented to the Society, December 31, 1928, March 29, 1929, and August 30, 1929; received by the editors October 11, 1929.

† L. E. J. Brouwer, *Beweis des Jordanschen Satzes für den n-dimensionalen Raum*, Mathematische Annalen, vol. 71 (1912), pp. 314–327.

‡ L. E. J. Brouwer, *Über Jordansche Mannigfaltigkeiten*, ibid., pp. 320–327.

§ J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, these Transactions, vol. 23 (1922), pp. 333–349. Familiarity with this work is assumed throughout the present paper.

The terminology "Betti number" is at present being employed more and more to denote connectivity numbers that are uniformly less by one unit than the numbers R^i of the present paper. However, since the present paper is based so much upon the above paper of Alexander, it seems advisable to retain the connectivity numbers as defined in that connection (although they are not termed "Betti numbers" by Alexander).

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and relation (2) states the important additional fact that every closed s-chain (where $s<2$) in the complement bounds an $(s+1)$-chain in the complement. Thus, although as Alexander pointed out in another connection* the domains complementary to a simple closed surface in E_3 may not be simply connected (may, in fact, require an infinite number of generators), every closed 1-chain in either of the domains bounds in that domain.

In contemplating so simple a surface as the torus, it becomes apparent that in formulating a *converse* of the Jordan-Brouwer separation theorem the relation (2) must be taken into account as well as relation (1). On the other hand, a glance at Brouwer's example cited above is sufficient to inform one that even with the duality relations (1) and (2), the accessibility of the points of a surface from each of its complementary domains is not enough to ensure that surface being, in E_3, a simple closed surface. Some condition must be imposed that will do away with the crinkliness of the surface. It occurred to the author that such a condition might be the following: If K is a simple closed surface in E_3, D one of the domains complementary to K, and ϵ a positive number, then there exists a positive number δ such that if Γ^0 is a closed 0-chain in D of diameter less than δ, then Γ^0 bounds an open 1-chain Γ^1 in D whose diameter is less than ϵ. This property of D may be concisely expressed by saying that *the 0-chains of D are uniformly homologous to zero in D.*

It will be shown in §1 of this paper (which may be considered as complementary to Alexander's paper on the Jordan-Brouwer separation theorem, since it utilizes the machinery introduced by him in order to treat accessibility properties into which he did not go), that the above condition is a necessary condition for the general n-dimensional case, and in §3 it will be shown that together with part of Alexander's duality relation it enables one to give, in E_3, a converse of the Jordan-Brouwer theorem.

The chief difficulty encountered by the author in applying the above, essentially combinatorial, properties,† to characterizing the sphere immersed in E_3, was that the known characterizations of the simple closed surface‡ were all of a topological type which did not seem to lend themselves very readily to association with the combinatorial properties of the complement. Accordingly a new characterization of the simple closed surface by means of internal properties that are readily associated with the combinatorial properties of the complement was worked out, and is given below in §2.

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* J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 8–10.

† It will be shown below, in the Appendix, that *uniformly homologous to zero* as a condition relative to the 0-chains of a domain is equivalent to the condition *uniform connectedness im kleinen* for the domain.

‡ The earliest characterization of which the author knows was announced by R. L. Moore and J. R. Kline, their abstract being in the Bulletin of the American Mathematical Society, vol. 28 (1922), p. 380. The same definition is given by Miss I. Gawehn (except for the greater generality of the space considered), in *Über unberandete 2-dimensionale Mannigfaltigkeiten*, Mathematische Annalen, vol. 98 (1927–28), pp. 321–354. Cf. also C. Kuratowski, *Une caractérisation topologique de la surface de la sphère*, Fundamenta Mathematicae, vol. 13 (1929), pp. 307–318, as well as abstracts by L. Zippin, Bulletin of the American Mathematical Society, vol. 35 (1929), p. 154 and p. 293.

In closing this introduction the author wishes to call to the attention of the reader the problem of giving a converse of the Jordan-Brouwer separation theorem in E_3 by certain accessibility conditions that are more in the spirit of the Schoenflies converse as formulated in E_2,[*] and it is hoped that if the attempt is made to find such conditions, the present work will be of some assistance as having already broken down some of the barriers. For in the opinion of the author the analysis situs relations between closed sets and their complements in spaces of three and higher dimensions will be most easily discovered by first having regard for the connectivity numbers of the complements,[†] and although it would be unwise to make any predictions, it would certainly seem as though to disregard these in the case of the $(n-1)$-sphere, when already known, in the attempt to obtain equivalent accessibility conditions, would involve an extreme waste of energy.

1. We shall show in this section[‡] that if M^{n-1} is an $(n-1)$-sphere im-

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[*] In this connection see J. R. Kline, *Separation theorems and their relation to recent developments in analysis situs*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 155–192. See especially pp. 156–159 and top of p. 191. As will be observed later on, *regulor* accessibility is an immediate consequence of the conditions given for the converse in the present paper.

[†] The recent work of P. Alexandroff bears out this opinion strikingly. Cf. his *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Annals of Mathematics, (2), vol. 30 (1928), pp. 101–187. The author wishes to seize this opportunity to express his indebtedness to Dr. Alexandroff, contact with whom, both through his memoirs and personally (during a recent brief visit at the University of Michigan), directed the attention of the author to the possibility of associating continuity and combinatorial methods in treating the problems of analysis situs.

[‡] The content of this section was presented, along with certain other results, to this Society, March 29, 1929, under separate title (cf. Bulletin of the American Mathematical Society, vol. 35, p. 458, abstract No. 22). It is felt, however, that the theorems and corollaries given above properly belong in the present paper. As for the other results of the paper just referred to, the theorem that the points of an i-cell $(0 \leqq i \leqq n-1)$ are regularly accessible from the complement in E_n is a special case of a more recent theorem established by the author, to the effect that in E_n the points of any closed set which is homeomorphic with a subset of E_{n-1} are regularly accessible from the complement; several of the other results were found to duplicate already known theorems and were withdrawn from the announcement in abstract. One of the things so deleted was an example which the author believes to be not without interest, and which he has not seen in the literature, viz., an arc in E_3 which cannot lie on, or bound, any set homeomorphic with a 2-cell, and which cannot be thrown into a straight line interval by a (1-1)-continuous transformation of E_3 into itself. (Cf. L. Antoine, *Sur l'homéomorphie de deux figures et de leurs voisinages*, Journal de Mathématiques, (8), vol. 4 (1921), pp. 221–325.) The chief interest of the example is its ease of description: On a great circle of the 2-sphere, let $ABCDEA$ be points in the cyclical order named, and let the portion BC of the

mersed in E_n,* and D is either of its complementary domains, then the 0-chains of D are uniformly homologous to zero in D. Consequently the domain D is uniformly connected im kleinen (see Appendix)—a fact which was established for the case where $n=2$ and M^{n-1} is a 1-sphere by R. L. Moore.† As a corollary it follows that the points of the sphere are not only accessible, but are regularly accessible, from the complementary domains.

THEOREM 1. *Let M^{n-1} be an $(n-1)$-sphere immersed in E_n, and let D be one of the domains of $E_n - M^{n-1}$. Then the 0-chains of D are uniformly homologous to zero in D.*

Let the domain $E_n-(M^{n-1}+D)$ be denoted by D_1.

Suppose the 0-chains in D are not uniformly homologous to zero. Then it easily follows that there is a point P on M^{n-1} and a spherical neighborhood U of P, such that in every neighborhood of P there is a 0-chain which does not bound in that part of D which lies in U. We shall denote the frontier of U by F.

There is a cell C^{n-1} of M^{n-1} which contains P as a non-boundary point and lies wholly in U. Denote the boundary of C^{n-1} by C^{n-2}, and let the set of all those points of M^{n-1} that are not interior points of C^{n-1} be denoted by B^{n-1}. Let V be a spherical neighborhood of P of such a radius that V contains no points of B^{n-1}. Let x_1^0 and x_2^0 be any two 0-cells of D that lie in V. Since the closed chain $x_1^0+x_2^0$ bounds a 1-chain L_B^1 interior to V, we have

$$x_1^0 + x_2^0 \sim 0 \qquad [\text{mod } 2,\ E_n - (F + B^{n-1})]. \tag{3}$$

In order to deal with a specific case, we may assume that a 0-cell, y_1^0, of L_B^1, lies in D_1, L_B^1 having been obtained, say, by joining x_1^0 and x_2^0 to such a cell by chains $L_{B_1}^1$ and $L_{B_2}^1$, respectively.

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arc $ABCD$ be replaced by a new portion containing two infinite sequences of simple trefoil knots (i.e., knots in the sense that the great circle becomes knotted), which of course have to leave the 2-sphere, and which respectively approach the points B and C as limiting points. This can be done in such a way that the new portion BC is still an arc from B to C that does not meet the arc $BAED$ of the great circle. Then the portions AB and CD of the great circle, together with the new portion BC, is an arc of the type described above.

[*] I.e., M^{n-1} is a point set which is in (1-1)-continuous correspondence with an $(n-1)$-sphere, and is therefore not necessarily formed by cells of the subdivisions of E_n which are used in the congruences and homologies of the proof. Cf. J. W. Alexander, these Transactions, loc. cit., for definitions of *chain, homology*, etc. The reader should be on his guard in the matter of terminology which has various meanings depending upon its application in a point-set theoretic sense or in a combinatorial sense. (E.g., "chain," "closed," "open," etc. Present usage seems to apply the term "chain" only to open chains, chains that are closed being termed "cycles." We are retaining the terminology of Alexander's paper, however, in the present connection.)

[†] R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–370.

On the other hand, as every 0-chain in D bounds in D, even though it may not bound im kleinen, there is a chain L_C^1 in D which is bounded by the chain $x_1^0+x_2^0$. Hence,

$$x_1^0 + x_2^0 \sim 0 \qquad (\text{mod } 2,\ E_n - C^{n-1}). \tag{4}$$

If now we can show that the closed chain

$$L_B^1 + L_C^1 \sim 0 \qquad (\text{mod } 2,\ E_n - C^{n-2}), \tag{5}$$

it will follow from relations (3), (4) and (5), and Alexander's Corollary W^i,* that $x_1^0+x_2^0$ bounds in the set common to D and U, and since no restriction is placed upon the choice of x_1^0 and x_2^0, a contradiction of the supposition that the 0-chains of D are not uniformly homologous to zero will be obtained. We therefore proceed to demonstrate the validity of relation (5).

Let C_B^{n-1} be an arbitrary cell of M^{n-1} interior to B^{n-1}. Then x_1^0 and y_1^0 bound a 1-chain K_1^1 in $E_n-(M^{n-1}-C_B^{n-1})$. The closed chain $K_1^1+L_{B1}^1$ links C^{n-2}, for if it did not, then, by virtue of Alexander's Corollary W^i, $x_1^0+y_1^0$ would bound in E_n-M^{n-1}. Similarly, the closed chain $K_1^1+L_C^1+L_{B2}^1$ links C^{n-2}. However, by relation (1) there is only one linearly independent non-bounding 1-chain in E_n-C^{n-2}. Hence

$$K_1^1 + L_{B1}^1 \sim K_1^1 + L_C^1 + L_{B2}^1 \qquad (\text{mod } 2,\ E - C^{n-2}). \tag{6}$$

Since homology (6) implies homology (5), the theorem is proved.

COROLLARY 1. *The points of an* $(n-1)$*-sphere,* M^{n-1}*, immersed in* E_n*, are regularly accessible*† *from the complement of the sphere.*

Let P be a point of M^{n-1}, and D one of the domains complementary to M^{n-1}. Since there are only two domains in the complement of M^{n-1}, it will be sufficient to prove P regularly accessible from D.

By virtue of Theorem 1 there exists (1) a sequence of spherical neighborhoods of P, $U_1, U_2, U_3, \cdots$, such that if r_k is the radius of U_k, then $r_k>r_{k+1}$, and $\lim_{k\to\infty} r_k=0$; and such that for $k>1$, any closed 0-chain of D that lies in U_k bounds a 1-chain that lies wholly in the common part of U_{k-1} and D; (2) a sequence of 0-cells $x_2^0, x_3^0, x_4^0, \cdots$, such that, for every k, x_k^0 lies in the common part of D and U_k.

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* These Transactions, loc. cit., p. 342.

† A point P is said to be *regularly accessible* from a point set D if for every positive number ϵ there exists a positive number δ such that if Q is a point of D whose distance from P is less than δ, then there is an arc PQ in $D+P$ whose diameter is less than ϵ. Cf. G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 650–657.

For any k, x_k^0 and x_{k+1}^0 bound a 1-chain, Γ_k^1, which lies wholly in the common part of D and U_{k-1}. As the set of points

$$M = \sum_{k=2}^{\infty}\Gamma_k^1 + P$$

is a continuous curve, there is an arc from x_2^0 to P that lies in M, and hence (except for P) in the common part of U_1 and D. As the first neighborhood, U_1, is arbitrary, and as in every neighborhood of a point x of D in U_2 there can be found a 0-cell x_2^0 in D, the corollary is proved.

If, in the proof of Theorem 1, M^{n-1} denotes an $(n-1)$-manifold, and P denotes a point which is interior to (i.e., not on the boundary of) a cell C^{n-1} which is one of the cells defining M^{n-1}, then the argument to show that $x_1^0+x_2^0$ is homologous to zero in $D\cdot U$ goes through as before. However, if P is such a point of the manifold that it cannot lie interior to any such cell, then the above 0-chain may fail to bound in $D\cdot U$. (Cf., for instance, the set K defined in the Appendix, paragraph beginning "Condition (3)".) Hence we can state the following theorem:

THEOREM 1a. *Let M^{n-1} be any $(n-1)$-manifold immersed in E_n, D a domain complementary to M^{n-1}, and P a point of M^{n-1} which can be considered as interior to an $(n-1)$-cell of definition of M^{n-1}. Then, if C^{n-1} is any such cell, and U is any spherical neighborhood of P that encloses only points of M^{n-1} that are interior to C^{n-1}, there exists a neighborhood V of P such that any* 0*-chain in $D\cdot V$ bounds a* 1*-chain in $D\cdot U$. If every point of M^{n-1} can be considered as lying interior to an $(n-1)$-cell of definition of M^{n-1}, the* 0*-chains of D are uniformly homologous to zero in D.**

COROLLARY 2. *The points of an $(n-1)$-manifold, M^{n-1}, immersed in E_n, are regularly accessible from the complement of the manifold.*

If P is an interior point of a cell C^{n-1} of M^{n-1}, the proof is the same as that of the above Corollary 1. If not, then P is on the boundaries of a certain number (at least four) of $(n-1)$-cells of M^{n-1}. Let V be any neighborhood of P. Since any $(n-1)$-cell of an $(n-1)$-manifold is homeomorphic with an $(n-2)$-sphere and its interior in E_{n-1}, there is a neighborhood U of P such that any point of M^{n-1} which lies in U and is interior to a cell C^{n-1} of M^{n-1}

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* In order to avoid confusion in the mind of the reader, it should perhaps be pointed out here that we are using the definition of manifold as given by Alexander in his paper on the Jordan-Brouwer Separation Theorem. That is, an i-manifold is a closed i-chain whose ith connectivity number has the value 2. Some authors (cf., for instance, Veblen's *Analysis Situs*, published by this Society, 1922, p. 88) state the additional condition that every point of the i-manifold shall have a neighborhood, relative to the manifold, which is an i-cell. The definition as given by Alexander of course yields greater generality to the results obtained.

is joined to P by an arc of $M^{n-1}\cdot V$ that lies, except for P, wholly interior to C^{n-1} (by Corollary 1). Let Q be any point of D (one of the domains of E_n-M^{n-1}) that lies in U. We shall show that there is an arc from P to Q lying, except for P, wholly in $D\cdot V$.

Let PQ be any arc of U from P to Q, and let P_1 be its first point on M^{n-1}, in the order from Q to P. Since in any neighborhood of a point of M^{n-1} there are points of M^{n-1} not on the boundaries of its $(n-1)$-cells, we may assume that P_1 is interior to a cell C^{n-1} of M^{n-1} whose boundary, C^{n-2}, contains P. There is an arc, t, of M^{n-1}, from P_1 to P and lying wholly interior to C^{n-1}, except for P, as well as interior to V. Let $P_2, P_3, P_4, \cdots$ be a sequence of points of t having P as a sequential limit point, and let ϵ denote the distance between t and the frontier of V as ordinarily defined for point sets.

Consider the portion P_1P_2 of t. Since all points of P_1P_2 are interior points of C^{n-1}, it is easy to show that there lies, in $D\cdot V$, an arc Q_1Q_2 approximating P_1P_2, where Q_1 is a point of QP_1 in the neighborhood of P_1, and Q_2 a point in the neighborhood of P_2. For if x is any point of P_1P_2, there is a number δ such that any 0-chain of D in $S(x, \delta)$ bounds a 1-chain of D in $S(x, \frac{1}{2}\epsilon)$ (Theorem 1a). A simple chain* of the neighborhoods $S(x, \delta)$ from P_1 to P_2, together with the joining of 0-cells in the successive links of the simple chain, yields the arc Q_1Q_2 desired.

By successive approximations to the arcs P_iP_{i+1} by arcs Q_iQ_{i+1} in $D\cdot V$ in such a manner that if x_i is any point of Q_iQ_{i+1}, $\lim_{i\to\infty} x_i=P$, it is clear that the set $QQ_1+\sum_{i=1}^{\infty} Q_iQ_{i+1}+P$ is a continuous curve containing an arc from Q to P and lying, except for P, wholly in $D\cdot V$.

2.† We shall now prove the following theorem:

THEOREM 2. *In a metric separable space let M be a compact set containing at least one simple closed curve, and satisfying the following conditions*: (1) *if t is an arc of M, then $M-t$ is connected*; (2) *if J is a simple closed curve of M, then $M-J$ is the sum of two uniformly connected im kleinen‡ components. Then M is a simple closed surface; i.e., a set homeomorphic with the sphere $x^2+y^2+z^2=1$.*§

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* Cf. R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 131–164. In particular, see bottom of p. 134, and Theorem 10 of this paper.

† The content of this section was presented to the Society December 31, 1928.

‡ A set of points K is said to be *uniformly connected im kleinen* if for every $\epsilon>0$ there exists a $\delta>0$, such that if P and Q are points of K whose distance apart is less than δ, then P and Q are in a connected subset of K whose diameter is less than ϵ. However, for an open subset of a continuous curve, it is easy to prove that the words "a connected subset" in this definition may be replaced by the words "an arc."

§ When this paper was in process of completion, Dr. Leo Zippin communicated to the author that he had succeeded in showing that a continuous curve C which has the properties that (1) if

For ease of reference, we shall divide the proof into sections:

I. *The set M is connected im kleinen.* For if J is a simple closed curve of M, and M_1 and M_2 are the components of $M-J$, then M_1+J and M_2+J are connected im kleinen by a theorem of R. L. Moore.* Accordingly the set M is itself connected im kleinen.

II. *If J is a simple closed curve of M, and M_1 and M_2 are the components of $M-J$, then J is identical with the boundary of each of the sets M_1, M_2.* For suppose J contains a point, P, which is not a limit point of M_1, say. Then an arc, t, of J, contains all the boundary points (if any) of M_1. Then

$$M - t = M_1 + (M_2 + J - t).$$

Since M_1 and M_2+J-t are mutually separated sets, a violation of condition (1) results.

III. *The set M is connected.* For M contains at least one simple closed curve, J, and, by II, J is the common boundary of the components of $M-J$.

IV. *If P is a point of M, then there is an arc of M which contains P, and of which P is not an end point.*†

Since M is a continuous curve, being a connected im kleinen, closed and connected set, there is an arc PP' of M, joining P to some point P' of $M-P$. By condition (1) the set $M-PP'=D$ is connected. If P is not a limit point of D, then there is a point Q of PP' distinct from both P and P', such that the portion PQ of PP' contains no limit points of D. Then the set

$$M - P'Q = D + (PQ - Q)$$

is not connected, since D and $PQ-Q$ are mutually separated sets. But this constitutes a violation of condition (1). Consequently D has P as a limit point.

The set $D+P$ is connected im kleinen. If not, then P is the only point at which it is not connected im kleinen, since D is an open subset of M. Then there exists, by Theorem 1 of my paper *Characterizations of continuous curves that are perfectly continuous*,‡ a positive number ϵ such that for every positive

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t is an arc of C, then $C-t$ is connected, (2) if J is a simple closed curve of C, then $C-J$ is not connected, is a simple closed surface. This definition is an obvious improvement over that given above, and by its use the last two parts of the proof of the theorem of §3 can be omitted.

* R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fundamenta Mathematicae, vol. 3 (1922), pp. 232–237, Theorem 1. The proof given by Moore evidently holds for a compact set in any topological space.

† Note added in proof: Since this paper was submitted to these Transactions, a paper by W. L. Ayres has appeared in the American Journal of Mathematics for October, 1929, in which it is shown (Theorem 10) that if a continuous curve in E_n contains no cut point, then the curve is cyclicly connected. As our curve M clearly contains no cut point by virtue of condition (1), it is obvious that the results IV and V follow from Ayres' theorem.

‡ Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 614–621.

number $\delta<\epsilon$ there is a point Q of D whose distance from P is less than δ and which is not in the same quasi-component with P of the set of points $(D+P)\cdot S(P,\ \epsilon)$.* It follows at once that there is a sequence of points $x_1, x_2, x_3, \cdots$, such that $\epsilon/2>\rho(x_n, P)>\epsilon/3$, and such that no two points of this sequence lie in the same quasi-component of $(D+P)\cdot S(P,\ \epsilon)$. As M is compact, there is a limit point, x, of the set of points $\sum_{n=1}^{\infty} x_n$. The point x cannot lie in D, since D is connected im kleinen. Hence x is on the arc PP', and since there is no loss of generality in assuming that $\rho(P, P')>\epsilon$, we can say that x is distinct from both P and P'.

However, D is uniformly connected im kleinen in the neighborhood of x. For let t be an arc in PP' whose end points are distinct from P and P', and which contains x. Since $M-t$ is connected, by condition (1), there exists in $M-t$, by a theorem of R. L. Moore,† an arc from P to P', and this arc together with the arc PP' contains a simple closed curve J which contains t, consisting of two arcs, AxB of PP' and AyB of $D+A+B$. Let the components of $M-J$ be M_1 and M_2, and let η be a positive number such that $S(x,\ \eta)$ contains no points of $J+PP'$ that are not points of t. Then by condition (2) there is a positive number ρ such that every two points of M_1 that lie in $S(x, \rho)$ are joined by an arc of M_1 in $S(x, \eta)$, and a similar statement holds for M_2. Since all points of $M-J$ in $S(P,\ \eta)$ belong to D, it is clear that if η is taken so small that $S(x,\ \eta)$ is a subset of $S(P,\ \epsilon)$, the points x_1, x_2, $x_3, \cdots$ cannot all lie in distinct quasi-components of $(D+P)\cdot S(P,\ \epsilon)$.

Accordingly $D+P$ must be connected im kleinen, and by a theorem of Whyburn‡ P is regularly accessible from D. Hence there is an arc of M containing P and of which P is not an end point.

V. *If P is a point of M, then P is on some simple closed curve of M.* This follows at once from IV, and the method used in IV to obtain the simple closed curve J.

VI. *If P is a point of M, $M-P$ is connected.* By V, P is on a simple closed curve J of M, and by condition (2) $M-J$ is the sum of two components which have, by II, J as common boundary. Since $M-P$ is the sum of these components together with some of their common boundary points, it is clear that $M-P$ is connected.

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* Hereafter, if P is a point and ϵ any positive number, the symbol $S(P,\ \epsilon)$ will denote a spherical neighborhood of P of radius ϵ; i.e., the set of all points of the space whose distance from P is less than ϵ. In accordance with the usual notation, $M\cdot N$ denotes the set of points common to M and N. By $\rho(P, Q)$ we denote the distance between two points P and Q.

† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem I. The theorem is evidently true for the case of a continuous curve in the general space we are considering.

‡ G. T. Whyburn, *Concerning accessibility in the plane and regular accessibility in n dimensions*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 504–510, Theorem 3. Although stated for E_n, the proof given for this theorem clearly holds in any metric space.

VII. *If M_1, M_2 and J are defined as in* II, *and AxB is an arc such that $<AxB>$*[*] *is in M_1 and $A+B$ on J, then $M_1- <AxB>$ is the sum of two components R_1 and R_2 whose boundaries are, respectively, $AxB+$an arc AyB of J, and $AxB+$an arc AzB of J, where AyB and AzB have only A and B in common.*

Let the two arcs into which A and B divide J be AyB and AzB. Let J_1 and J_2 be simple closed curves defined as follows:

$$J_1 = AxB + AyB,$$
$$J_2 = AxB + AzB.$$

By condition (2), $M-J_1$ is the sum of two components R_1 and R_1' whose common boundary, by II, is J_1. Let R_1 be that one of these components which lies in M_1. (It is clear that M_2 lies in one of the components of $M-J_1$, say in R_1', and that R_1' must therefore contain $<AzB>$, thus requiring that R_1 be a subset of M_1.) Let that one of the components of $M-J_2$ that lies in M_1 be denoted by R_2.

Suppose that M_1 contains a point P that is not in the set $R_1+R_2+<AxB>$. Then P lies in R_1', and if P_1 is a point of M_2, there is an arc PP_1, from P to P_1, in R_1'. If Q is the first point of J on this arc, in the order from P to P_1, then $PQ-Q$ lies in M_1. It is clear that Q is in $<AzB>$; let $S(Q, \epsilon)$ be a neighborhood of Q that does not enclose any point of AxB. There is a neighborhood $S(Q, \delta)$ such that any two points of $M_1 \cdot S(Q, \delta)$ are joined in $S(Q, \epsilon)$ by an arc of M_1. Since points of $PQ-Q$ and R_2 both lie in $S(Q, \delta)$, and such points are then joined by connected subsets of $M-J_2$, a contradiction results from the supposition that M_1 contains points not in the set $R_1+R_2+<AxB>$. Hence $M_1=R_1+R_2+<AxB>$.

VIII. *If P is a point of M, and ϵ is any positive number, there is a simple closed curve J in M which does not contain P and such that both J and the component of $M-J$ which contains P lie in $S(P, \epsilon)$.*

By V, P is on some simple closed curve J of M. Denote the components of $M-J$ by M_1 and M_2. We can, of course, suppose that J is not wholly in $S(P, \epsilon)$, and that both M_1 and M_2 have points in the exterior of $S(P, \epsilon)$.

By virtue of the uniform connectedness im kleinen of M_1 and M_2, together with II, it is easy to show the existence, in $S(P, \epsilon)$, of arcs A_1xB_1 and A_2yB_2, where $<A_1xB_1>$ and $<A_2yB_2>$ lie in M_1 and M_2, respectively, and the points A_1, A_2, B_1, B_2 lie on J in the order $A_1A_2PB_2B_1A_1$, and such that the arc A_1PB_1 of J lies in $S(P, \epsilon)$. Let the simple closed curve $A_1xB_1+B_1B_2+A_2yB_2$

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[*] By $<AxB>$ we denote the set $AxB-A-B$.

$+A_1A_2$ (where A_1A_2 and B_1B_2 are portions of A_1PB_1 on J) be denoted by J_1. Let that component of $M-J_1$ that contains P be denoted by R_1.

By VII, $M_1-<A_1xB_1>$ is the sum of two components, one of which is bounded by the simple closed curve $T_1=A_1PB_1+A_1xB_1$; denote this component by K_1. Similarly, that component of M_2 bounded by the simple closed curve $T_2=A_2PB_2+A_2yB_2$ denote by K_2. Then

$$R_1 = K_1 + K_2 + < A_2PB_2 > .$$

For by VII, $R_1-<A_2PB_2>$ is the sum of two components H_1 and H_2, bounded, respectively, by T_1 and T_2. Now T_1 bounds only two components, one of which contains M_2. If H_1 were the latter component, then H_1 would contain $<A_2yB_2>$, which is clearly impossible. Then H_1 is that component of $M-T_1$ that does not contain M_2, i.e., K_1. Similarly, $H_2=K_2$. We note that if α is a point of M_1 that is not in K_1+T_1, then R_1 does not contain α.

If R_1 does not lie wholly in $S(P, \epsilon)$, there is a point, P_1, in R_1, such that $\rho(P, P_1)=\epsilon$.

In a similar manner, we can show the existence, in $S(P, \epsilon/2)$, of arcs $C_1x_1D_1$ and $C_2y_2D_2$, where $<C_1x_1D_1>$ and $<C_2y_2D_2>$ lie in K_1 and K_2, respectively, and the end points of these arcs lie on A_2PB_2 in the order $A_2C_1C_2PD_2D_1B_2$. Denote the simple closed curve formed by these two arcs, together with the portions C_1C_2 and D_1D_2 of A_2PB_2, by J_2. Denote the component of $M-J_2$ that contains P by R_2.

By VII, $K_1-<C_1x_1D_1>$ is the sum of two components, one of which, N_1, is bounded by the simple closed curve $F_1=C_1x_1D_1+C_1PD_1$; similarly, that component of $K_2-<C_2y_2D_2>$ bounded by the curve $F_2=C_2y_2D_2+C_2PD_2$ denote by N_2. Then we can show, as above, that

$$R_2 = N_1 + N_2 + < C_2PD_2 > .$$

Consequently, R_2 is a subset of R_1, and does not contain the point α defined above.

We may continue in this way indefinitely, obtaining simple closed curves J_3, J_4, J_5, $\cdots$, with corresponding components R_3, R_4, R_5, $\cdots$, all containing P, but not α, and each component a subset of the preceding; also, requiring that for each positive integer n, the curve J_n lie in $S(P, \epsilon/n)$.

For some n, R_n is a subset of $S(P, \epsilon)$. For if not, there is for every n a point P_n of R_n such that $\rho(P, P_n)=\epsilon$. Since M is compact, the set of points $\sum_{n=1}^{\infty} P_n$ has at least one limit point Q, which is in every set R_n and such that $\rho(P, Q)=\epsilon$. Common to all the sets R_1, R_2, R_3, $\cdots$, there is a continuum C, which contains P and Q.

By VI, $M-P$ is connected, and therefore contains an arc t with end points Q and α.* There exists a positive integer k such that ϵ/k is less than the distance between P and the arc t. Then the set $C+t$ is a connected subset of $M-J_k$, since C lies in R_k and t cannot meet J_k. However, P lies in R_k and α lies in $M-(J_k+R_k)$. Thus the supposition that for no n does R_n lie in $S(P, \epsilon)$ leads to a contradiction.

The conclusion of Theorem 2 now follows in either one of the following ways: (1) by virtue of the Moore-Kline-Gawehn† definition of the simple closed surface, whose conditions are now seen to be fulfilled; or (2) by virtue of R. L. Moore's Axioms Σ_1 for plane analysis situs.‡ For it is easy to see, in view of what has been shown above, that if Q is an arbitrary point of M, then $M-Q$ is a topological plane, in that it satisfies the axioms Σ_1. Thus, if J is a simple closed curve of $M-Q$, let that component of $M-J$ which does not contain Q be called a region.

3.§ We shall prove the following theorem:

THEOREM 3 (Converse of the Jordan-Brouwer Separation Theorem in E_3). *Let K be a closed and bounded set in E_3, such that $E_3-K=S_1+S_2$, where S_1 and S_2 are mutually exclusive and*

(1) *every arc from a point of S_1 to a point of S_2 contains at least one point of K;*

(2) *if P is a point of K and Q a point not in K, then in every neighborhood of P there is a point P' such that there is an arc from P' to Q lying except for possibly P' wholly in E_3-K;*

(3) *the 0-chains of $S_i (i=1, 2)$ are uniformly homologous to zero in S_i; and*

(4) *the Betti number* (mod 2) $R^1(E_3-K)=1$.

Then K is a simple closed surface.

We shall show that the set K satisfies all the conditions of Theorem 2.

I. *The set K is connected.* For if not, it is the sum of two mutually separated sets, K_1 and K_2. Let P_i be a point of K_i $(i=1, 2)$. By a theorem of Knaster and Kuratowski,¶ there exists a continuum C in E_3-K which separates P_1 and P_2. The continuum C lies wholly in one of the sets S_1, S_2.

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* Cf. R. L. Moore, Mathematische Zeitschrift, loc. cit.

† Cf. Bulletin abstract of Moore and Kline, loc. cit., and I. Gawehn, loc. cit.

‡ R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 131–164. See also R. L. Wilder, *Concerning R. L. Moore's axioms Σ_1 for plane analysis situs*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 752–760.

§ The contents of this section were presented to the Society August 30, 1929.

¶ B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255. The theorem referred to here is Theorem 37, p. 233. This theorem holds true in E_n, since the theorem of Brouwer used in its proof holds in E_n, for both bounded and unbounded sets.

For if it contains points of both S_1 and S_2, then S_1, say, contains a limit point of S_2. But then, since K is closed, this limit point would be joined to a point of S_2 by an arc that does not meet K, a violation of condition (1). Then C lies in S_1, say.

Now in the neighborhood of any point P of K there lie points of both S_1 and S_2. For if we let U be any neighborhood of P and x a point of S_1, say, there is, by condition (2), a point P' in U which is joined to x by an arc $P'x$ and such that $P'x-P'$ lies in E_3-K. It is clear that $P'x-P'$ must lie in S_1, since x is in S_1; and as U must contain points of $P'x-P'$, there is a point of S_1 in U.

Accordingly, in the same domain complementary to C that contains P_1 there is a point x_1 of S_2. By condition (2) there is, in the same domain complementary to C as P_2, a point x_2 such that there is an arc x_1x_2 which lies, except possibly for x_2, wholly in E_3-K. Then $x_1x_2-x_2$ lies in S_2, and there is no point of C on x_1x_2, since C is in S_1. But x_1 and x_2 must be separated by C in E_3. Thus the supposition that K is not connected leads to a contradiction.

II. *The set K is connected im kleinen.* For if not, it follows from a theorem of R. L. Moore* that there exist two concentric spheres, R_1 and R_2, and a sequence of subcontinua of K, namely M_∞, M_1, M_2, M_3, $\cdots$, such that (1) each of these continua contains at least one point of R_1 and R_2, respectively, but no point exterior to R_1 or interior to R_2, (2) no two of these subcontinua have a point in common, and no two of them contain points of any connected subset of K that lies wholly in R_1+R_2+I, where I is the annular domain bounded by R_1 and R_2, (3) M_∞ is the sequential limiting set of the sequence of continua M_1, M_2, M_3, $\cdots$.

Let P be a point of M_∞ in I, and let R be a spherical neighborhood of P such that R' lies in I (where R' denotes R together with its boundary). Let us consider R as a space, $\overline{E}_3$, and if M is a point set, let us denote the product $M\cdot R$ by $\overline{M}$.

There exists a positive integer m such that $\overline{M}_i$, for $i\geqq m$, is non-vacuous. The set $\overline{K}$ is closed relative to $\overline{E}_3$, and by condition (2) of Moore's theorem just quoted, no connected subset of $\overline{K}$ joins $\overline{M}_i$ and $\overline{M}_j$ ($i\neq j$; i, $j\geqq m$). There exists a spherical neighborhood, U, of P in $\overline{E}_3$, such that any 0-chain of S_i in U is homologous to zero in $\overline{S}_i$. Let i and j be values such that $\overline{M}_i$

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* R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–370. The theorem referred to here is not given any explicit statement, in theorem form, in this paper, but will be found in such form in the same author's *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 296–297.

and $\overline{M}_j$ have points P_i and P_j, respectively, in U. There exists a continuum, C (continuum relative to $\overline{E}_3$) that lies in $\overline{E}_3 - \overline{K}$ and separates P_i and P_j in $\overline{E}_3$. For there is a separation of $\overline{K}$ into two mutually separated sets containing $\overline{M}_i$ and $\overline{M}_j$, respectively,* and the theorem of Knaster and Kuratowski used in I applies in $\overline{E}_3$.

Now, as shown in a similar case in I, C must lie wholly in, say, S_1. But in every neighborhood of P_i, and of P_j, there are points of S_2; in particular, there are such points in neighborhoods of P_i and P_j that lie in $U - C \cdot U$, and these bound 1-chains of $\overline{S}_2$. As such chains must contain points of C, and hence of S_1, a contradiction is established, and K must be connected im kleinen.

III. *The sets S_1 and S_2 are connected.* Consider S_1, and let x and y be any two of its points. Let P be any point of K, and let U_1 be a spherical neighborhood of P. By condition (3) there is a spherical neighborhood U_2 of P such that if a^0 and b^0 are two 0-cells of S_1 in U_2, then a^0 and b^0 bound a 1-chain of S_1 in U_1. Now, as already shown, U_2 contains two points, x_1 and y_1, which are joined to x and y, respectively, by arcs xx_1 and yy_1 in S_1. There is no loss of generality in assuming that x_1 and y_1 are 0-cells of the subdivision of E_3, and accordingly bound a 1-chain L^1 of S_1 in U_1. Then $xx_1 + L^1 + yy_1$ is a connected subset of S_1 joining x and y, and therefore S_1 is connected.

IV. *No arc of K disconnects K.* For let t be an arc of K and suppose that $K - t = K_1 + K_2$, where K_1 and K_2 are mutually separated sets. The sets $K_1 + t$ and $K_2 + t$ are closed.

Let P and Q be points of S_1 and S_2, respectively, which are also 0-cells of the subdivision of E_3. By condition (2), $K_1 + t$ does not separate P and Q in E_3; let L_1^1 be a 1-chain bounded by P and Q in the complement of $K_1 + t$. Similarly, let L_2^1 be a 1-chain bounded by P and Q in the complement of $K_2 + t$. Since the closed 1-chain $L_1^1 + L_2^1$ cannot link the arc t,† it follows from a theorem of J. W. Alexander† that $P + Q$ bounds a 1-chain in $E_3 - K$; in other words, condition (1) is violated. Hence t does not separate K.

* V. *If J is a simple closed curve of K, then $K - J$ is not connected.* Suppose that $K - J$ is connected.

By the Alexander duality relation (see introduction) there is a closed 1-chain Γ^1 which links J. We shall first show that Γ^1 has points in both S_1 and S_2.

Suppose Γ^1 is contained in $S_1 + K$. Then it is clear that Γ^1 contains points of K, for if not, by condition (4) we should have

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* This follows from the fact that the closed set $K \cdot R'$ is separated between $M_i \cdot R'$ and $M_j \cdot R'$.

† J. W. Alexander, these Transactions, loc. cit.

*IV'. There is at least one simple closed curve in K.
For by IV, K has no cut-point and hence is cyclicly connected.
(Cf. footnote †, p. 639).

$$\Gamma^1 \sim 0 \qquad (\text{mod } 2, E_3 - J),$$

which is impossible since Γ^1 links J. Let the distance between J and Γ^1, as ordinarily defined for point sets, be denoted by ϵ. By condition (3) of the theorem, there is a number δ_ϵ such that if Γ^0 is a closed 0-chain of S_1 of diameter less than δ_ϵ, then there is a 1-chain Γ_1^1 such that

$$\Gamma_1^1 \equiv \Gamma^0 \qquad (\text{mod } 2, S_1)$$

where the diameter of Γ_1^1 is less than $\frac{1}{4}\epsilon$.

We may assume, without loss of generality, that the 1-chain Γ^1 is irreducible, and that its 1-cells have a given cyclic order. We may also assume that the one-cells of Γ^1 are all of diameter less than $\frac{1}{4}\delta_\epsilon$.

The cells of Γ^1 may be divided into two classes, according to whether their boundaries lie wholly in S_1 or not. We may start with a given cell c^1, and consider the successive cells of Γ^1 in their cyclic order. If both end-cells of c^1 lie in S_1, then c^1 may be replaced by a 1-chain K^1 which has the same boundary, but which lies entirely in S_1 and has a diameter less than $\frac{1}{4}\epsilon$. We may agree to say that the cell c^1 has been transformed into the *chain-cell* K^1. Proceeding to the next 1-cell of Γ^1, which we may denote by d^1, suppose that one of its end-cells is on K. Denote the cell that it has in common with c^1 by a^0, and the cell on K by b^0. There is a 0-cell, $\bar{b}^0$, in S_1, whose distance from b^0 is less than $\frac{1}{4}\delta_\epsilon$, and since the distance from a^0 to $\bar{b}^0$ is less than δ_ϵ, these two cells bound a 1-chain K_1^1 in S_1, whose diameter is less than $\frac{1}{4}\epsilon$. We shall replace b^0 by $\bar{b}^0$, and d^1 by K_1^1, and say that b^0 and d^1 have been transformed, respectively, into $\bar{b}^0$ and K_1^1. Suppose the next cell on Γ^1 is e^1, and that its end-cell distinct from b^0, viz. c^0, is also on K. There is a 0-cell $\bar{c}^0$ in S_1 whose distance from c^0 is less than $\frac{1}{4}\delta_\epsilon$. Clearly the distance from $\bar{b}^0$ to $\bar{c}^0$ is less than δ_ϵ, and hence these two cells bound a 1-chain K_2^1 in S_1 of diameter less than $\frac{1}{4}\epsilon$. We shall let $\bar{c}^0$ and K_2^1 be the transforms, respectively, of c^0 and e^1.

Proceeding through all of the cells of Γ^1 as just indicated, we replace each cell of Γ^1 by its transform, and obtain a new closed 1-chain, Γ_1^1, which we may call the transform of Γ^1. Now Γ_1^1 lies wholly in S_1, and accordingly, by application of condition (4),

$$\text{(a)} \qquad \Gamma_1^1 \sim 0 \qquad (\text{mod } 2, E_3 - J).$$

We shall show that relation (a) is impossible.

We note that if c^1 is any 1-cell of Γ^1, then every point of the transform, K^1, of c^1, is at a distance less than $\frac{1}{2}\epsilon$ from either end-cell of c^1. That is, both of the chains c^1 and K^1 lie within a spherical neighborhood T of radius ϵ which has its center at one end-cell of c^1 and accordingly contains no point of

J. By introducing, if necessary, two new 1-chains, each bounded by an end-cell of c^1 and its transform, a closed 1-chain L^1 is obtained in T which contains c^1 and K^1 and such that

$$L^1 \sim 0 \qquad (\text{mod } 2,\, T,\, E_3 - J). \tag{b}$$

By adding homology (a) and all homologies of type (b), we have

$$\Gamma_1{}^1 + \sum L^1 \sim 0 \qquad (\text{mod } 2,\, E_3 - J). \tag{c}$$

However, we have that, adding modulo 2,

$$\Gamma_1{}^1 + \sum L^1 = \Gamma^1 \tag{d}$$

(it can be assumed that only one chain is introduced between each 0-cell of Γ^1 and its transform). Combining relations (c) and (d) we have

$$\Gamma^1 \sim 0 \qquad (\text{mod } 2,\, E_3 - J).$$

This contradicts the fact that Γ^1 links J. Hence Γ^1 must have points in both S_1 and S_2.

We shall now show that the supposition that $K-J$ is connected is inconsistent with the fact that Γ^1 links J, and has points in both S_1 and S_2. It will first be necessary to make a transformation of Γ^1.

Let ϵ_1 be a positive number less than $\frac{1}{4}\epsilon$. Since K is uniformly connected im kleinen (being a bounded, connected im kleinen continuum), there is a positive number ϵ_2, such that if P and Q are points of K at a distance apart less than ϵ_2, then P and Q are joined by an arc of K of diameter less than ϵ_1. Let δ_{ϵ_2} be a positive number such that if P and Q are 0-cells of S_i $(i=1 \text{ or } 2)$ at a distance apart less than δ_{ϵ_2}, then P and Q bound a 1-chain in S_i of diameter less than $\frac{1}{4}\epsilon_2$. We may assume that the 1-cells of Γ^1 are all of diameter less than $\frac{1}{4}\delta_{\epsilon_2}$.

The transformation that we now effect on Γ^1 is very similar to the one described above, except that the new constants just defined are involved. Supposing that we start with a cell c^1 of Γ^1 whose end-cells are both in S_1, we replace it by a chain-cell K^1 with the same boundary and diameter less than $\frac{1}{4}\epsilon_2$. If the next cell b^1 has an end-cell on K, we transform it into a chain-cell in S_1. Of course we ultimately come to a cell c^1 one of whose end-cells (the one last affected by a transformation), a^0, is in S_1+K, and the other, b^0, is in S_2. (Indeed, if Γ^1 had none of its bounding 0-cells in S_2, the whole chain Γ^1 could be transformed into a chain in S_1, just as in the above proof, and a contradiction obtained as before. Similarly, if we encounter a case of one bounding 0-cell of Γ^1 in S_2 (S_1) and the neighboring 0-cells in S_1 (S_2), the 1-cells which they bound may be transformed into chain-cells lying in

S_1 (S_2).) It is unnecessary to use condition (3) here, since any 1-chain K^1 of diameter less than $\frac{1}{4}\epsilon_2$ and bounded by $\bar{a}^0$ and b^0 will do for the transform of c^1 ($\bar{a}^0$ being the transform of a^0). The next 0-cell, c^0, is in S_2 (else we would have proceeded as indicated in the above parenthesis) and the transformation to be effected is obvious. The subsequent transformations should also be obvious, since we now proceed from S_2 in just such a manner as we proceeded from S_1, viz., by not actually "crossing" into S_1 until two successive bounding 0-cells are encountered in S_1.

The outcome of these transformations on the cells of Γ^1 is to obtain a new 1-chain, which we shall still denote by Γ^1 and which still links J (this can be shown as above), but which has the following properties: (1) its intersections with K occur on certain chain-cells, $K_1^1, K_2^1, \cdots, K_n^1$ in the order named; we shall henceforth call these the *crossing-cells* of Γ^1; (2) if K_i^1 is any crossing-cell, the diameter of K_i^1 is less than $\frac{1}{4}\epsilon_2$; (3) the boundary cells of K_i^1 lie in S_1 and S_2, respectively; (4) if K_i^1 and K_{i+1}^1 are bounded by the cells a^0, b^0, c^0 and d^0 (these occurring in the order named), then the cells a^0 and d^0 are both in $S_1(S_2)$; in other words, if one crossing-cell leads from S_1 to S_2, the next leads from S_2 to S_1. The last property is very important, and shows that there is an even number of crossing-cells.

We shall now proceed to replace Γ^1 by a 1-chain which still links J but has only *two* crossing-cells.

Starting with K_1^1, suppose that the bounding 0-cells of K_1^1 and K_2^1 occur in the order a^0, b^0, c^0, d^0, where a^0 is in S_1. In S_1, let A_1^1 be a new 1-chain bounded by a^0 and d^0, and in S_2 let A_2^1 be the 1-chain of Γ^1 which lies in S_2 and is bounded by b^0 and c^0. Denote the portion of Γ^1 from d^0 to a^0 by B^1. Now we cannot have both of the following relations:

$$\text{(e)} \qquad A_1^1 + K_1^1 + A_2^1 + K_2^1 \sim 0,$$

$$\text{(f)} \qquad A_1^1 + B^1 \sim 0 \qquad (\text{mod } 2,\ E_3 - J),$$

since the sum of these homologies would imply the homology

$$\Gamma^1 \sim 0 \qquad (\text{mod } 2,\ E_3 - J).$$

If (e) fails to hold, we have secured the type of 1-chain linking J that we set out to obtain, viz., a chain with only two crossing-cells. If (e) holds, we can proceed with the 1-chain

$$\Gamma^1 + K_1^1 + A_2^1 + K_2^1 + A_1^1 = A_1^1 + B^1 \qquad (\text{mod } 2)$$

which has two less crossing-cells than Γ^1, viz., $K_3^1, \cdots, K_n^1$, by next considering the crossing-cells K_3^1 and K_4^1.

In any case, we eventually arrive at a closed 1-chain which links J, and has only two of the original crossing-cells of Γ^1. We shall continue to call this chain Γ^1, and we shall suppose its crossing-cells are K_1^1 and K_2^1, their boundary cells being denoted, as before, by a^0, b^0, c^0, d^0. Denote the portion of Γ^1 lying in S_1 and bounded by a^0+d^0 by L_1^1 and the portion in S_2 bounded by b^0+c^0 by L_2^1.

Because of the way in which the above transformation was carried out, at least one of the boundary cells of K_i^1 $(i=1, 2)$ is identical with a cell of the original chain Γ^1, so that a spherical neighborhood T_i of diameter ϵ_2 about K_i^1 certainly encloses no points of J. We may assume that K_1^1 and K_2^1 are homeomorphic with simple arcs, and on K_i^1 $(i=1, 2)$ let x_i and y_i be points of K such that the portions a^0x_1 (d^0x_2) and b^0y_1 (c^0y_2) of K_1^1 (K_2^1) contain no points of K other than x_1 and y_1 (x_2 and y_2).

Since x_1 and x_2 lie in $K-J$, and $K-J$ is connected, there is an arc x_1x_2 in $K-J$.* Denote the distance between x_1x_2 and J by ϵ_3, let ϵ_4 denote a positive number less than both ϵ_1 and ϵ_3, and let δ_{ϵ_4} be a positive number such that if two 0-cells of S_i are at a distance apart less than δ_{ϵ_4}, they bound a 1-chain in S_i of diameter less than $\frac{1}{4}\epsilon_4$ (condition (3)).

Let $F_1, F_2, \cdots, F_m$ be points on the arc x_1x_2, where $F_1=x_1$ and $F_m=x_2$, occurring in the order named, and so selected that

$$\delta(F_iF_{i+1}) < \tfrac{1}{4}\delta_{\epsilon_4} \qquad (i = 1, 2, \cdots, m-1),$$

where F_iF_{i+1} denotes the portion of x_1x_2 between F_i and F_{i+1}. Let $P_1, P_2, \cdots, P_m$ be a set of 0-cells in S_1 such that P_1 is on the portion a^0x_1 of K_1^1 and P_m on the portion d^0x_2 of K_2^1, and the following relations hold:

$$\rho(P_i, F_i) < \tfrac{1}{4}\epsilon_4,$$
$$\rho(P_i, P_{i+1}) < \delta_{\epsilon_4}.$$

According to the definition of δ_{ϵ_4}, there exists for each i an open 1-chain H_i^1 in S_1 bounded by P_i and P_{i+1}, and such that $\delta(H_i^1)<\frac{1}{4}\epsilon_4$. We define $C_1^1=\sum_{i=1}^{m-1} H_i^1$, and we also note that every point of H_i^1 lies within a distance $\frac{1}{2}\epsilon_4$ of F_i.

Let T_i^1 $(i=1, 2)$ be a spherical neighborhood concentric with T_i and of diameter $2\epsilon_1$. Clearly T_i^1 contains no point of J. According to the definition of ϵ_2, and because of the fact that the distance from x_i to y_i is less than ϵ_2 (the diameter of K_i^1 being less than ϵ_2), there is an arc t_i in K which lies wholly in T_i and has end points x_i and y_i. We can approximate the arc t_1 by a $\frac{1}{4}\delta_{\epsilon_4}$-chain† of 0-cells in S_2, with first point, z_1, on the portion y_1b^0 of K_1^1, and last point, z_h, and such that not only is every point of the chain at a distance less than $\frac{1}{4}\epsilon_4$ from some point of t_1, but in particular $\rho(z_h, F_1) < \frac{1}{4}\epsilon_4$. Then, in a manner similar to that used in obtaining C_1^1, we obtain a 1-chain B_1^1 bounded by $z_1 + z_h$, lying wholly in S_2, and such that every point of B_1^1 is within a distance $\frac{1}{2}\epsilon_4$ from some point of t_1. It is clear that a spherical neighborhood T_1^2 concentric with T_1 and of diameter $3\epsilon_1$, will contain B_1^1, and enclose no points of J.

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* R. L. Moore, Mathematische Zeitschrift, loc. cit.

† The word "chain" is here used in the ordinary point-set theoretic sense; i.e., if ϵ is a positive number, and P and Q are points, an ϵ-chain from P to Q is a set of points $P_1, P_2, \cdots, P_n$, where $P_1=P$ and $P_n=Q$, and $\rho(P_i, P_{i+1})<\epsilon$ $(i=1, 2, \cdots, n-1)$.

We can obtain similarly a 1-chain B_2^1 with reference to t_2, so that the boundary of B_2^1 is a point w_1 on the portion y_2c^0 of K_2^1 and a point w_k in S_2 whose distance from F_m is $< \frac{1}{4}\epsilon_4$. Let T_2^2 be a spherical neighborhood analogous to T_1^2.

Let $Q_1, Q_2, \cdots, Q_m$ be points in S_2 chosen with reference to the points $\{F_i\}$ just as the points $\{P_i\}$ were chosen in S_1, and so that $Q_1 = z_h$ and $Q_m = w_k$. On these points can be built up a 1-chain B_3^1 of S_2, bounded by Q_1 and Q_m, and approximating the arc x_1x_2 in a manner similar to that in which C_1^1 approximates x_1x_2.

We now define the following 1-chains:

$$
\begin{aligned}
C_2^1 &= \sum_{i=1}^{3} B_i^1, \\
\Gamma_1^1 &= L_1^1 + a^0P_1 + C_1^1 + P_md^0, \\
\Gamma_2^1 &= C_1^1 + P_1z_1 + C_2^1 + w_1P_m, \\
\Gamma_3^1 &= C_2^1 + z_1b^0 + L_2^1 + c^0w_1,
\end{aligned}
$$

where a^0P_1, P_md^0, etc., are portions of K_1^1 and K_2^1. We note that

$$\text{(g)} \qquad \Gamma^1 = \sum_{i=1}^{3} \Gamma_i^1 \qquad \pmod{2}.$$

The following homologies follow from condition (4) of the theorem:

$$\text{(h)} \qquad \Gamma_1^1 \sim 0,$$

$$\text{(i)} \qquad \Gamma_3^1 \sim 0 \qquad (\text{mod } 2, E_3 - J).$$

Consequently, since Γ^1 links J, it follows from relations (g), (h) and (i) that Γ_2^1 links J.

If we join P_1 and z_h by a 1-chain B_4^1 every point of which is at a distance less than $\frac{1}{4}\epsilon_4$ from x_1 $(=F_1)$ and P_m and w_k similarly by a 1-chain B_5^1, it is easy to see that since the following relations hold:

$$
\begin{aligned}
B_1^1 + B_4^1 + P_1z_1 &\sim 0 \qquad (T_2^1, E_3 - J), \\
B_2^1 + B_5^1 + P_mw_1 &\sim 0 \qquad (T_2^2, E_3 - J),
\end{aligned}
$$

the closed chain

$$\Gamma = C_1^1 + \sum_{i=3}^{5} B_i^1$$

links J.

However, this is impossible for the following reasons. The points P_i and Q_i ($i=2, 3, \cdots, m-1$) bound a 1-chain N_i^1, every point of which is at a distance less than $\frac{1}{4}\epsilon_4$ from F_i. For convenience of notation, let $N_1^1 = B_4^1$ and $N_m^1 = B_5^1$. Let U_i ($i=1, 2, \cdots, m$) be a spherical neighborhood of F_i of radius ϵ_4. Then every point of the closed chain

$$E_i^1 = H_i^1 + N_i^1 + \overline{H}_i^1 + N_{i+1}^1,$$

where $\overline{H}_i^1$ is the portion of B_3^1 from Q_i to Q_{i+1}, lies in U_i. Hence

$$\text{(j)} \qquad E_i^1 \sim 0 \qquad (\text{mod } 2,\ E_3 - J).$$

But

$$\Gamma = \sum_{i=1}^{m} E_i^1 \qquad (\text{mod } 2),$$

and by adding all homologies (j), we get

$$\Gamma \sim 0 \qquad (\text{mod } 2,\ E_3 - J).$$

Thus, on the assumption that $K-J$ is connected, we are able to find a closed chain Γ, homologous to the chain Γ^1 which links J, but which does not itself link J. As this is impossible, $K-J$ cannot be connected.

VI. *If J is a simple closed curve of K, then $K-J$ contains at most two components.* For suppose $K-J$ contains at least 3 components, M_1, M_2, and M_3. Since K is a continuous curve, each component of $K-J$ is also a quasi-component of $K-J$,* and it is therefore easy to see, with the result of section IV, that each component of $K-J$ has all of J as its boundary.

There exists on J a set of points occurring in the following order, $P_1A_1Q_1B_1P_2B_2Q_2A_2P_1$, and such that there is an arc $A_ix_iB_i$ ($i=1, 2$) lying, except for A_i and B_i, wholly in M_i. Define a simple closed curve J_1 as follows:

$$J_1 = \sum_{i=1}^{2} A_ix_iB_i + A_1P_1A_2 + B_1P_2B_2,$$

where $A_1P_1A_2$ and $B_1P_2B_2$ are arcs of J. Then the set $K-J_1$ is connected, in contradiction to the result of section V. For if we denote the set

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* Cf. R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11 (1928), pp. 127–131.

$K-(M_1+M_2+J)$ by R, it is clear that the set $R_1=R+\sum_{i=1}^{2}<A_iQ_iB_i>$ is connected. Also, the set $M_1-<A_1x_1B_1>$ contains only components that have boundary points on either $<A_1Q_1B_1>$ or $<A_2Q_2B_2>$. For suppose $M_1-<A_1x_1B_1>$ contains a component C with boundary points only on the set $t=A_2P_1A_1x_1B_1P_2B_2$. Then the set

$$K - t = C + \left[(M_1 - <A_1x_1B_1> - C) + M_2 + R + \sum_{i=1}^{2} < A_iQ_iB_i >\right]$$

is not connected, contrary to the result of IV. In a like manner it is shown that all components of $M_2-A_2x_2B_2$ have limit points on $\sum_{i=1}^{2}<A_iQ_iB_i>$, and it is clear that the set

$$K - J_1 = R_1 + \sum_{i=1}^{2}(M_i - A_ix_iB_i)$$

is connected.

VII. *Denoting the two components of $K-J$ by M_1 and M_2, respectively, the sets M_1 and M_2 are uniformly connected im kleinen.* For suppose M_1 is not uniformly connected im kleinen. Then there is a point P on J, a spherical neighborhood R of P, and a sequence of pairs of points of M_1, $\{P_n, Q_n\}$, such that $\lim_{n\to\infty}P_n=P$ and $\lim_{n\to\infty}Q_n=P$, and for no n are P_n and Q_n joined in R by any connected subset of M_1. Denote the set $M_i\cdot R$ by $\overline{M}_i$ $(i=1, 2)$, and the frontier of R by F.

Denote by C the component of $K\cdot R$ determined by P. Let R_1 be a spherical neighborhood of P of smaller radius than R, which encloses no points of J that are not on the arc, t, component of $J\cdot R$ determined by P. We shall denote $J\cdot R$ by t_1.

Let R_2 be another spherical neighborhood of P such that any point of K within R_2 is joined to P by an arc of K that lies in R_1 and such that R_2 encloses no points of $K-C$. Let P_k, Q_k be a pair of points of the sequence $\{P_n, Q_n\}$ lying in R_2, and let the component of $\overline{M}_1$ determined by P_k be denoted by H_1; let $\overline{M}_1-H_1=H_2$. Then H_1 and H_2 are mutually separated.

In R_2, let x_1 and x_2 be points of S_1 and S_2, respectively, that are 0-cells of the subdivision of E_3.

(a) The set C separates R between x_1 and x_2. For x_1 and x_2 bound a chain Γ_1^1 in $E_3-[F+(K-C)]$, and if they bound a chain Γ_2^1 in $E_3-(F+C)$, then, since the closed chain $\Gamma_1^1+\Gamma_2^1$ is homologous to zero in R, it cannot link the product $[F+(K-C)]\cdot[F+C]$, which is a subset of F, and accordingly x_1+x_2 is homologous to zero in E_3-K (Alexander's Corollary W_i, loc. cit.). This is a contradiction of condition (1) of the theorem.

(b) Under the assumption that M_1 is not uniformly connected im kleinen in the neighborhood of P, C does not separate x_1 and x_2 in R. Let y_1 and y_2 be the first points of K on a straight line from x_1 to x_2, in the orders x_1 to x_2 and x_2 to x_1, respectively. There exist arcs y_1P_k, y_1Q_k, y_2P_k and y_2Q_k of K in R_1. By the use of the usual approximation based on condition (3) of the theorem, we obtain a closed 1-chain, Γ^1, consisting of two open chains Γ_1^1 and Γ_2^1 bounded by x_1 and x_2, such that Γ^1 lies in R_1, and cuts K only in certain arbitrarily small neighborhoods of P_k and P_j, respectively. The following congruences hold:

$$\Gamma_1^1 \equiv x_1 + x_2 \qquad [\text{mod } 2,\, E_3 - (F + H_1')],$$

$$\Gamma_2^1 \equiv x_1 + x_2 \qquad [\text{mod } 2,\, E_3 - (F + H_2' + \overline{M}_2')]$$

(where H_1' denotes H_1 together with all its limit points, etc.). We may assume that the product of H_1' and H_2' is a subset of t_1', so that if we can show that Γ^1 bounds in $E_3-(F+H_1')\cdot(F+H_2'+\overline{M}_2')$ or, since the latter set contains $E_3-(F+t_1)$, if we can show that Γ^1 bounds in $E_3-(F+t_1)$, we shall have that x_1 and x_2 are not separated in R by C (Alexander's Corollary W_i, loc. cit.).

Since M_1 is connected, it can be shown, by the methods employed in section V, that Γ^1 does not link J. Consequently, there exists a chain Γ_1^2 such that

$$\Gamma_1^2 \equiv \Gamma^1 \qquad (\text{mod } 2,\, E_3 - J,\, E_3 - t').$$

But since Γ^1 lies in R_1, and the latter contains no points of $F+J-t$, we have

$$\Gamma_2^2 \equiv \Gamma^1 \qquad [\text{mod } 2,\, E_3 - (F + J - t)].$$

Now since the product of $F+J-t$ and t' is just two points, and neither Γ_1^2 nor Γ_2^2 meets the arc $J-t$ joining these points, we have that

$$\Gamma_1^2 + \Gamma_2^2 \sim 0 \quad [\text{mod } 2,\, E_3 - t'\cdot(F + J - t)].$$

Hence, by Alexander's Corollary W_i (loc. cit.),

$$\Gamma^1 \sim 0 \qquad [\text{mod } 2,\, E_3 - (F + t_1)].$$

As this is the relation we wished to prove, in order to show that the 0-chain x_1+x_2 bounds in $R-C$, we have shown that x_1 and x_2 are not separated by C in R.

As (a) and (b) are in contradiction, the assumption that M_1 is not uniformly connected im kleinen cannot hold. A like statement of course holds for M_2, and the theorem is proved.

The following theorem follows immediately from Theorem 3, from which it is only slightly different:

THEOREM 4 (Converse of the Jordan-Brouwer Separation Theorem in E_3). *Let K be a closed and bounded set in E_3, such that* (1) *the Betti numbers* (mod 2) $R^0(E_3-K)$ *and* $R^1(E_3-K)$ *are respectively equal to* 2 *and* 1, (2) *if D is a component of E_3-K, the* 0*-chains in D are uniformly homologous to zero in D, and every point of K is a limit point of D. Then K is a simple closed surface.*

Since there are two components, S_1 and S_2, in E_3-K, condition (1) of Theorem 3 is satisfied. Since every point of K is a limit point of each of these components, condition (2) of Theorem 3 is satisfied; for, it is to be noted, condition (2) of Theorem 3 allows P' to lie in that complementary domain which contains Q. Conditions (3) and (4) of Theorem 3 are restated in Theorem 4. Thus Theorem 4 follows at once from Theorem 3.

4. **Appendix.** The independence of the conditions of Theorem 3 is established as follows:

Condition (1): In E_3, using rectangular coördinates, let K be the set of points $(x, y, 0)$ such that $x^2+y^2 \leqq 1$. Let S_1 be the set of all points for which $z>0$; let $S_2=E_3-(K+S_1)$.

Condition (2): Using spherical coördinates, let K be the set of all points (ρ, ϕ, θ) for which $1 \leqq \rho \leqq 2$, let S_1 be the set of points for which $\rho<1$, and S_2 the set for which $\rho>2$.

Condition (3): Using spherical coördinates, let K' denote the set of points $\rho=1$. On K', let t denote the arc consisting of points $(1, 0, \theta)$ such that $0 \leqq \theta \leqq \pi$. On t, if P denotes any point $(1, 0, \theta)$, let P' denote the point $(1, 0, \pi-\theta)$. The set K is obtained by continuously deforming K' so that each point P coincides with P', but points not on t remain distinct as before. The complement of K is two domains, neither of which satisfies condition (3) of Theorem 3.

Condition (4): The torus.

The equivalence, in the case of open sets in E_n, of the conditions *uniformly homologous to zero* and *uniformly connected im kleinen*, is established in the following theorem:

THEOREM 5. *In order that an open set G in E_n should be uniformly connected im kleinen, it is necessary and sufficient that the* 0*-chains in G should be uniformly homologous to zero in G.*

The condition is necessary. Let ϵ be any positive number. Since G is uniformly connected im kleinen, there exists a positive number δ such that if P and Q are 0-cells of G whose distance apart is less than δ, there is an arc PQ of G whose diameter is less than $\frac{1}{4}\epsilon$.

Let $E_n - G = F$, and denote the distance between F and the arc PQ by η_1. Let η be a positive number smaller than either of the numbers η_1, $\frac{1}{2}\epsilon$. Let $P_1(=P)$, $P_2, \cdots, P_n(=Q)$ be points of PQ such that $\rho(P_i, P_{i+1}) < \frac{1}{4}\eta (i=1, 2, \cdots, n-1)$. The subdivision of E_n may be extended so that there exist 0-cells $F_1(=P_1)$, $F_2, \cdots, F_n(=P_n)$ such that $\rho(F_i, F_{i+1}) < \frac{1}{4}\eta$ and $\rho(F_i, P_i) < \frac{1}{4}\eta$. Let T_i be a spherical neighborhood of P_i of radius $\frac{1}{2}\eta$. Then clearly F_{i+1} lies in T_i, and consequently $F_i + F_{i+1}$ bounds a 1-chain H_i in T_i. Since no point of F lies in T_i, $H_i \cdot F = 0$. Consequently the chain $K^1 = \sum^{n-1}_{=1} H_i$ lies in G.

The diameter of K^1 is less than ϵ. For let x_1 and x_2 be points of K^1. Since x_1 lies in some sphere T_i, and hence $\rho(x_1, P_i) < \frac{1}{2}\eta < \frac{1}{4}\epsilon$, and since there exists, similarly, a P_j such that $\rho(x_2, P_j) < \frac{1}{4}\epsilon$, it follows at once from the fact that the diameter of PQ is $< \frac{1}{4}\epsilon$, that $\rho(x_1, x_2) < \epsilon$.

The condition is sufficient. If ϵ is a positive number, there is a positive number δ such that if x_1^0 and x_2^0 are 0-cells of G such that $\rho(x_1^0, x_2^0) < \delta$, then $x_1^0 + x_2^0$ bounds a chain K^1 of G of diameter $< \frac{1}{4}\epsilon$. Let P_1 and P_2 be any two points of G such that $\rho(P_1, P_2) < \frac{1}{4}\delta$. The subdivision of E_n may be extended so that there exist 0-cells y_1^0 and y_2^0 such that $\rho(x_i^0, P_i) < \frac{1}{4}\delta$ $(i=1, 2)$, and such that there are arcs y_iP_i of diameter $< \frac{1}{4}\epsilon$ in G. Since $\rho(y_1^0, y_2^0) < \delta$, $y_1^0 + y_2^0$ bounds a chain K^1 of G of diameter $< \frac{1}{4}\epsilon$. The set $K^1 + \sum_{i=1}^{2} y_iP_i$ contains an arc from P_1 to P_2 of diameter $< \epsilon$.

As a result of Therem 5 we can restate Theorem 4 as follows:

THEOREM 4′(Converse of the Jordan-Brouwer Separation Theorem in E_3). *In E_3, the common boundary, K, of two uniformly connected im kleinen domains, one of which is bounded, is a simple closed surface, provided that the Betti number* (mod 2) $R^1(E_3 - K) = 1$.

The problem in n dimensions, $n > 3$. Regarding a converse of the Jordan-Brouwer Separation Theorem in E_n, where $n > 3$, the author will not hazard any guesses here, but merely indicate the likelihood that by an extension of the conditions of Theorem 4, such a converse may be obtained. Thus, condition (1) of Theorem 4 may be replaced, as a result of the Alexander duality theorem, by the condition that the Betti numbers $R^0(E_3 - K)$ and $R^i(E_3 - K)$ $(i=1, 2, \cdots, n-2)$ shall be respectively equal to 2 and 1, and condition (2) may be replaced by the statement that the closed i-chains $(i=0, 1, \cdots, n-2)$ of D are uniformly homologous to zero in D, retaining, of course, the statement that every point of K is a limit point of D. The validity of such a condition (2) is established by the following theorems, with which we close the present paper. (The extension of the definition of "uniformly homologous to zero" to i-chains where $i > 0$ should be obvious.)

THEOREM 6. *Let M^{n-1} be an $(n-1)$-sphere immersed in E_n, and let D be one of the domains of $E_n - M^{n-1}$. Then the closed i-chains $(n>i>0)$ of D are uniformly homologous to zero in D.*

Suppose the conclusion of the theorem untrue. Then there is a point P of M^{n-1} and a contracting sequence of i-chains of D, $K_1^i, K_2^i, K_3^i, \cdots$, such that P is the sequential limit point of any set of points $P_1, P_2, P_3, \cdots$, where P_n, for every n, is a point of K_n^i, and such that there is a positive number ϵ for which K_n^i, for every n, fails to bound any $(i+1)$-chain of D of diameter less than ϵ.

Let $S(P, \epsilon/2)$ be a spherical neighborhood of P of diameter $\epsilon/2$. Then there is an $(n-1)$-cell, C^{n-1}, of M^{n-1}, which contains P as an interior point and lies wholly in $S(P, \epsilon/2)$. Let R denote a spherical neighborhood of P which encloses only points of M^{n-1} that lie interior to C^{n-1}. Denote the boundaries of $S(P, \epsilon/2)$ and C^{n-1} by $F(P, \epsilon/2)$ and C^{n-2}, respectively. The set C^{n-2} is of course an $(n-2)$-sphere immersed in E_n.

Let us first consider the case where $i<n-1$.

There is a number j, such that K_j^i lies wholly in R. Then, denoting by B^{n-1} the set of all points of M^{n-1} that are not interior points of C^{n-1}, we have the following congruences:

$$L_1^{i+1} \equiv K_j^i \qquad (\text{mod } 2, E_n - C^{n-1}),$$
$$L_2^{i+1} \equiv K_j^i \qquad \{\text{mod } 2, R, E_n - [B^{n-1} + F(P, \epsilon/2)]\}.$$

Since $i>0$, $i+1$ is greater than 1, and since by Alexander's Theorem $X^\ddagger$ an $(n-2)$-sphere immersed in E_n can be linked only by a 1-chain, the following homology holds:

$$L_1^{i+1} + L_2^{i+1} \sim 0 \qquad (\text{mod } 2, E_n - C^{n-2}).$$

As C^{n-2} is the common part of C^{n-1} and $B^{n-1}+F(P, \epsilon/2)$, it follows from Alexander's Corollary $W^\ddagger$ that

$$K_j^i \sim 0 \qquad [\text{mod } 2, D\cdot S(P, \epsilon/2)].$$

As this contradicts our assumption, the theorem is proved for $i<n-1$.

If $i=n-1$, the proof is trivial. Each of the chains K_j^i may be assumed irreducible (i.e., $R^i(K_j^i)=2$), and thus, by Alexander's Theorem Y, separates E_n into just two domains, one of which, D_1, must lie in R. The domain D_1 is an open n-chain bounded by K_j^i and containing no point of M^{n-1}.

A similar proof, together with the result of Theorem 1a, yields the following general result:

THEOREM 7. *Let M^{n-1} be any $(n-1)$-manifold immersed in E_n, D a domain complementary to M^{n-1}, and P a point of M^{n-1} which can be considered as* *interior to an* $(n-1)$*-cell of definition of* M^{n-1}. *Then if* U *is a neighborhood of* P *bounded by an* $(n-1)$*-sphere and enclosing only points of* M^{n-1} *that are interior to* C^{n-1}, *there exists, in* U, *a neighborhood* V *of* P *such that any i-chain* $(i=0, 1, \cdots, n-1)$ *that lies in* $D \cdot V$ *bounds an* $(i+1)$*-chain of* $D \cdot U$. *If* M^{n-1} *is of the type such that every one of its points can be considered as lying interior to an* $(n-1)$*-cell of definition of the manifold, then the i-chains* $(i=0, 1, \cdots, n-1)$ *of* D *are uniformly homologous to zero in* D.

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ON A CERTAIN TYPE OF CONNECTED SET WHICH CUTS THE PLANE

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It has been shown* by J. R. Kline that a continuum† which contains more than one point, and which remains connected upon the removal of any connected subset, is a simple closed curve (Jordan curve)‡. It has been shown§ by C. Kuratowski that if no subcontinuum of a bounded continuum C cuts|| C, then C is a simple closed curve.

The purpose of the present paper is to investigate some of the properties of a plane connected set M which contains more than one point, and which remains connected upon the omission of any connected subset. It should be noted that neither of the words "bounded" and "closed" is used in this characterization of M. In fact, M may be unbounded, punctiforme¶, or both. In any case, it will be shown in §1 that if A and B are any two distinct points of M then M is the sum of two sets, M_1 and M_2, which are irreducibly connected** from A to B and such that $M_1-(A+B)$ and $M_2-(A+B)$ are mutually separated††. It will also be shown in §3 that a set M having these properties always cuts the plane.

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*J. R. Kline, *Closed and connected sets which remain connected upon the removal of certain connected subsets*, Fundamenta Mathematicae, V (1924), pp. 3-10.

†A point set is said to be *closed* if it contains all its limit points. A point set is said to be *connected* if, no matter how it be divided into two subsets, at least one of these contains a limit point of the other. A *continuum* is a closed and connected point set.

‡If A and B are distinct points, an *arc* from A to B is a bounded continuum which contains A and B and which is disconnected by the omission of any one of its points which is distinct from A and B. A and B are called the *end-points* of the arc, and any other point of the arc is called an *inner* point. A *simple closed curve* is the sum of two arcs which have in common only their end-points. A point set M is *bounded* if there exists a circle K such that every point of M lies interior to K.

§C. Kuratowski, *Contribution à l'étude de continus de Jordan*, Fundamenta Mathematicae, V (1924) pp. 112-122.

||If C is a continuum and N is a subset of C, then N is said to *cut* C provided there exist at least two points A and B of the set $C-N$ which do not lie in a subcontinuum of $C-N$.

¶A set is said to be *punctiforme* if it contains no subcontinuum which contains more than one point.

**A set K is said to be *irreducibly connected* from A to B if K is a connected set containing A and B and no proper connected subset of K contains both A and B. (A set N is a *proper* subset of set K provided N is a subset of K and $K-N$ is not vacuous).

††Two point sets M and N are said to be *mutually exclusive* if they have no point in common. If M and N are mutually exclusive and neither contains a limit point of the other, then they are said to be *mutually separated*.

In §2 of this paper is a collection of theorems which I have found necessary to the demonstration of the theorem of §3.

§1. *Theorem* 1. *Let M be a connected point-set such that if N is any connected subset of M, $M-N$ is connected, and let A and B be any two distinct points of M. Then M is the sum of two sets, M_1 and M_2, which are irreducibly connected from A to B, and such that $M_1-(A+B)$ and $M_2-(A+B)$ are mutually separated sets.*

Proof. $M-A$ is connected by hypothesis. The set $M-(A+B)$ is not connected. For if $M-(A+B)$ were connected, then $M-[M-(A+B)]=A+B$ would be connected, which is absurd.

Since $M-(A+B)$ is not connected,

$$M-(A+B)=K+L,$$

where K and L are mutually separated sets.

As $M-A$ is a connected set which is disconnected by the omission of one point, B, it follows* that $K+B$ and $L+B$ are connected sets. Similarly, since $M-B$ is a connected set which is disconnected by the omission of a point A, $K+A$ and $L+B$ are connected sets. Then the sets $K+A+B$ and $L+A+B$ are connected as each is the sum of two connected sets having points in common.

Let

$$K+A+B=M_1,\quad L+A+B=M_2.$$

M_1 and M_2 are irreducibly connected from A to B. For suppose that M_1 is not irreducibly connected from A to B. Then there exists a subset, X, of K, such that M_1-X is connected and contains A and B. By hypothesis, the set $M-(M_1-X)$ must be connected. But

$$M-(M_1-X)=X+L$$

and X and L are mutually separated since K and L are mutually separated. Hence the supposition that M_1 is not irreducibly connected from A to B leads to a contradiction. Similarly, it can be shown that M_2 is irreducibly connected from A to B. Then M is the sum of two sets, M_1 and M_2, which are irreducibly connected from A to B, and such that $M_1-(A+B)$ $[=K]$ and $M_2-(A+B)$ $[=L]$ are mutually separated sets.

Theorem 2. *Let M be a connected set such that if N is any connected subset of M, $M-N$ is strongly connected*†. *Then M is a simple closed curve.*

Proof. By Theorem 1, if A and B are two distinct points of M,

$$M=M_1+M_2,\quad M_1=K+A+B,\quad M_2=L+A+B,$$

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*Cf. B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, II (1921), pp. 206-255, Th. VI.

†A point set K is *strongly* connected if every two points of K lie in a subcontinuum of K.

where M_1 and M_2 are irreducibly connected from A to B and K and L are mutually separated.

K and L are connected[*]. Since

$$M - L = M_1,$$

M_1 is strongly connected, and A and B lie in a subcontinuum, N, of M_1. As every subcontinuum of a set irreducibly connected between two points is an arc[†], N is an arc. Clearly M_1 and N are identical, as otherwise M_1 could not be irreducibly connected from A to B. Hence M_1 is an arc from A to B. Similarly, M_2 is an arc from A to B.

Since M is the sum of two arcs which have only their end-points in common, it is a simple closed curve.

§2. C. Carathéodory has made[‡] an admirable analysis of the structure of the boundary of a simply connected domain. In all that follows, I shall assume familiarity with his work, as well as with the work of Miss Marie Torhorst in the same connection[§].

Theorem A. *For every interval of prime ends*[||] *of a simply connected domain G there exist infinitely many distinct points of the boundary of G that are accessible for prime ends of that interval*[¶].

Proof. Let E be a prime end of the interval, I, distinct[**] from the ends of the interval[††] I. Let P be a principal point[‡‡] of E. Then E can be defined by a chain of cross-cuts[§§], $q_1, q_2, q_3, \ldots$ which lie on concentric circles that converge to their common centre[||||]. There exists a positive integer, n, such that the prime ends corresponding to[¶¶] q_i $(i \geqq n)$ are prime ends of I. For every value of i, let E_i be one of the prime ends corresponding to q_i, and P_i the end-point of q_i in E_i. The points $P_n, P_{n+1}, P_{n+2}, \ldots$ are distinct points satisfying the theorem.

By an argument similar to that used above the following theorem can be established:

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[*] Cf. B. Knaster and C. Kuratowski, *loc. cit.*, Corollaire XXIV.

[†] Cf. B. Knaster and C. Kuratowski, *loc. cit.*, Corollaire XXVIII.

[‡] C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen 73 (1913), pp. 323-370.

[§] Marie Torhorst, *Über den Ränder der einfach zusammenhängender Gebiete*, Mathematische Zeitschrift, 6 (1921), pp. 44-65.

[||] *Primenden*.

[¶] It should be noted that this is not the same as stating that every interval of prime ends contains infinitely many prime ends of the first or second kind. This theorem states that every interval of prime ends contains infinitely many prime ends of the first or second kinds whose accessible points are distinct.

[**] When two prime ends are not identical, I shall call them *distinct*.

[††] By the *ends of an interval of prime ends* I mean the prime end which precedes all others in the interval and the prime end which is preceded by all others in the interval.

[‡‡] *Hauptpunkt*.

[§§] *Querschnittskette*.

[||||] Cf. C. Carathéodory, *loc. cit.*, Satz. VIII.

[¶¶] Cf. M. Torhorst, *loc. cit.*, p. 49, §§ 26 and 27.

Theorem A′. *Every prime end, E, of a simply connected domain G is the limit of a convergent sequence* of prime ends of G of the first or second kind, whose accessible points are all distinct. If P is a principal point of E, then the prime ends of this sequence can be so chosen that their accessible points have P as a sequential limit point*†.

Definition. Let G be a simply connected domain, M a set of points of G, P a point of the boundary of G, and E a prime end of G that contains P. Then P is a *limit point of M for the prime end E* provided there exists a sequence of points of M, $P_1, P_2, P_3, \ldots$, which converges‡ to the prime end E and has P as a sequential limit point.

Theorem B. *Let M be a subset of a simply connected domain, G, and let P be a limit point of M on the boundary of G. Then there exists a prime end of G, E_P, which contains P, and such that P is a limit point of M for the prime end E_P.*

Proof. Since P is a limit point of M, there exists a sequence of points of M, $P_1, P_2, P_3, \ldots$, which has P as a sequential limit point.

Let G be mapped conformally by means of an analytic function, $f(z)$, on the interior of the unit-circle, $|z|<1$. For every value of n, let the image of P_n in $|z|<1$ be x_n. Then the set of points $x_1, x_2, x_3, \ldots$, has at least one limit point, x, on the unit circle $|z|=1$.

As x is a limit point of $x_1, x_2, x_3, \ldots$, there exists a subsequence of this sequence, $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$, which has x as a sequential limit point. Let E_P be that prime end of G which corresponds to x§. Then the sequence of points $P_{n_1}, P_{n_2}, P_{n_3}, \ldots$, converges to the prime end E_P, and since P is a sequential limit point of this sequence it follows that P is a limit point of M for the prime end E_P.

Theorem C. *Let E be a prime end of a simply connected domain G, and P a point contained in E. Then the chain of crosscuts $q_1, q_2, q_3, \ldots$, which determines E can be so chosen that for any positive integer, n, there exists a circle, K, with centre at P, such that no point of q_n lies either interior to, or on, K.*

Proof. E can be defined by a chain of cross-cuts, $q_1, q_2, q_3, \ldots$, which lie on concentric circles that converge to their common centre. If P is the common centre of these circles, the circle, K, of which q_{n+1} is a portion, satisfies the theorem. If P is not the common centre of these circles, there exists a positive number, N, such that for $i>N$, the circle of which q_i is a portion neither encloses nor contains P. Let k be a positive integer greater than N. Then the chain of cross-cuts $q_k, q_{k+1}, q_{k+2}, \ldots$, and a circle K with centre P and radius less than $\delta(P, C_k)$|| (where C_k is the circle of which q_k is a portion) satisfy the theorem.

Theorem D. *Let A, B, C, D be four distinct points of the boundary, β, of a simply connected domain G, accessible, respectively, for the prime ends E_A, E_B, E_C, E_D,*

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*Cf. C. Carathéodory, *loc. cit.*, p. 351, footnote.

†A point P is a *sequential limit point* of a sequence of points $P_1, P_2, P_3, \ldots$, provided that if, K is any circle with centre at P there exists a positive integer n such that for $i>n$, P_i lies interior to K.

‡Cf. C. Carathéodory, *loc. cit.*, § 11, III.

§Cf. C. Carathéodory, *loc. cit.*, Satz XIII.

|| M and N being point-sets, $\delta(M, N)$ denotes the lower limit of all the distances xy where x is any point of M and y is any point of N.

where E_A *and* E_B *separate* E_C *and* E_D. *Let* AB *and* CD *be arcs which lie, except for their end-points,* A, B, *and* C, D, *entirely in* G, *and such that the prime ends corresponding to* AB *are* E_A *and* E_B, *and the prime ends corresponding to* CD *are* E_C *and* E_D. *Let* t *be an arc with end-points* A *and* B *and containing no point of* CD. *Under these conditions, there exists a simple closed curve,* J, *which consists of a sub-arc of* t *and a sub-arc of* AB, *and such that* C *is interior to* J *and* D *exterior to* J, *or vice versa.*

Proof. The arc CD separates G into two domains, G_1 and G_2*, such that E_A and E_B are contained in G_1 and G_2, respectively. Denote the boundary of G_1 by β_1 and the boundary of G_2 by β_2.

Let

$$AB \times G_1 + A = N_1, \quad AB \times G_2 + B = N_2.$$

On t, in the order from A to B, there exists a last point, x, of N_1. To show this, add to N_1 all its limit points, and call the resulting set $\overline{N}_1$†. Since the latter set is closed, there will exist on t, in the order from A to B, a last point of $\overline{N}_1$. Since the only limit points of N_1 that do not belong to N_1 lie on CD, this point must be a point of N_1, by virtue of the fact that t contains no points of CD.

After x, on t, in the order from x to B, let y be the first point of N_2. The existence of y can be established by an argument similar to that used to show the existence of x. The points x and y are distinct; for $x = A$ or x is a point of G_1, and therefore x is not a limit point of N_2, or a point of N_2.

That portion‡ of t from x to y is an arc t_1. That portion of AB from x to y is an arc t_2; t_1 and t_2 have only x and y in common, hence their sum is a simple closed curve J.

On t_1, in the order from x to y, let x' be the first point of β_1. On t_1, in the order from y to x, let y' be the first point of β_2; t_2, together with those portions of t_1 from x to x' and from y to y', forms an arc $x'y'$ which lies, except for x' and y', wholly in G. Let the prime ends of G corresponding to $x'y'$ be $E_{x'}$ and $E_{y'}$, where $x'y'$ converges to x' in $E_{x'}$ and to y' in $E_{y'}$.

$E_{x'}$ and $E_{y'}$ are separated by E_C and E_D, since $E_{x'}$ and $E_{y'}$ are contained in G_1 and G_2, respectively. Therefore there will exist, on CD, in the order from C to D, a first point, w, of $x'y'$, and in the order from D to C a first point z of $x'y'$. Between C and w, and between D and z on CD there can be no points of J, since $J - x'y'$ is a subset of t.

Hence if C and D are both interior (or exterior) to J, all points of the arcs Cw and Dz (subsets of CD), except w and z, are interior (or exterior) to J. No

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*Cf. C. Carathéodory, *loc. cit.*, §§ 5-7.

†If M is any point-set, I shall hereafter in this paper use the symbol $\overline{M}$ to denote M together with its limit points.

‡If S is an ordered set of elements, and x and y are elements of S, that portion of S from x to y is the set of all elements $[e]$ such that $x = e$, or $y = e$, or e is between x and y.

point of that portion of t_2 from w to z is a limit point of the rest of J, except w and z, and all such points lie in G. Hence there exist, on Cw and Dz, points w' and z', respectively, which can be joined by an arc $w'z'$ that contains no point of $\beta+J$, and no point of Cw or Dz except w' and z'. The sum of the arcs Cw' (subset of CD), $w'z'$, and $z'D$ (subset of CD), is an arc which joins C and D, lies wholly in G, except for C and D, and converges to the prime ends E_C and E_D, but contains no point of the arc $x'y'$. This is impossible, since E_C and E_D are separated by $E_{x'}$ and $E_{y'}$. It follows that C and D cannot both be interior (or exterior) to J.

Corollary. If A and B separate C and D on a simple closed curve K, AB and CD are arcs joining A, B, and C, D, respectively, and lying, except for their end-points, interior to K, and t is an arc from A to B that contains no point of CD, then there exists a simple closed curve J consisting of a sub-arc of AB and a sub-arc of t, such that C is interior to J and D exterior to J, or vice versa.

The notion of *prime part* of a continuum has been recently introduced by Hans Hahn[*]. If P is a point of a continuum M, the prime part of M which contains P is the set of all points $[x]$ belonging to M such that for every positive number e there exists a finite set of irregular[†] points, $P_1, P_2, P_3, \ldots, P_n$, of M such that the distances $PP_1, P_1P_2, P_2P_3, \ldots, P_nx$ are all less than e. A bounded continuum M is a *simple continuous arc of prime parts* provided there exist two distinct prime parts of M, p and q, such that M is disconnected by the omission of any one of its prime parts which is distinct from p and q[‡]. The prime parts p and q are called the *extremities* of the arc.

Theorem E. *Let E be a prime end of a simply connected domain, G, and let P be any point of G. Then there exists a chain of cross-cuts, $q_1, q_2, q_3, \ldots$, defining E, and a simple continuous arc, t, of prime parts such that (i) the extremities of t are P and ω, the set of principal points of E, (ii) $t-\omega$ is a subset of G, and converges to E, (iii) every prime part of t, except ω, is a point, (iv) for every n, q_n and t have only one point in common.*

Proof. It is, of course, only necessary to show that this theorem is satisfied for some particular point of G and a particular choice of the cross-cuts defining E, since then it easily follows that the theorem is satisfied for all points of G.

Carathéodory shows[§] the existence of a domain G', subset of G, and a prime end E' of G' which is contained in E and contains only principal points of E. He then shows[‖] that the image in G of that radius of the unit circle (upon the interior of which G' has been mapped conformally) which joins the centre to the image of E' is a curve c' which begins in a point P of G' and converges to E' and to E. The only points of the boundary of G that are limit points of c'

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[*] Hans Hahn, *Über irreduzible Kontinua*, Wiener Berichte, 130 (1921), pp. 217-250.

[†] An *irregular* point of M is a point at which M is not connected im kleinen. Cf. H. Hahn, Wiener Berichte, 123 (1914) (2433).

[‡] For a more general definition of simple continuous arc of prime parts, see R. L. Moore, *Concerning the prime parts of certain continua which separate the plane*, Proc. Nat. Acad. Sci., 10 (1924), pp. 170-175.

[§] *loc. cit.*, p. 361, Satz XIX.

[‖] *loc. cit.*, p. 362, § 43.

are the points of ω, and every one of these is a limit point of c'. Let $c'+\omega=c$. The curve c is a simple continuous arc of prime parts whose extremities are P and ω. If E is a prime end of the first or second kind, c is of course a simple continuous arc in the ordinary sense. If E is a prime end of the third or fourth kind, then ω is a prime part of c. This is easily shown as follows: ω being a continuum*, and c clearly connected im kleinen at all points except those belonging to ω, it is only necessary to show that c is not connected im kleinen at any point of ω. Suppose c were connected im kleinen at a point x of ω. Let y be a point of ω distinct from x. Let K_1 be a circle with centre x and radius less than $\delta(x, y)$. Then there exists a circle K_2 concentric with K_1 and such that if z is any point of c interior to K_2, z can be joined to x by an arc zx of c which lies wholly interior to K_1. Let z be a point of $c-\omega$, and let z' be the first point of ω on zx in the order from z to x. That portion of zx from z to z' is an arc zz' which lies, except for z', wholly in G. The prime end corresponding to zz' is E, since every subset of c which has a limit point on the boundary of G converges to E. Then z' is accessible for E and $\omega=z'$. Thus the supposition that c is not connected im kleinen at x leads to a contradiction.

The proof that every prime part of t except ω, is a point, and that t is a simple continuous arc of prime parts, is only dependent on the fact that the conformal mapping of G' upon the interior of the unit circle establishes a one-to-one continuous correspondence between the points of G' and the points interior to the unit circle.

It remains to show that the chain of cross-cuts defining E can be so chosen as to satisfy (iv). It will, however, be simpler to first choose the chain of cross-cuts defining E and then modify the simple continuous arc of prime parts c. Let x be any point of ω and let $q_1, q_2, q_3, \ldots$, be a chain of cross-cuts (defining E) which consist of portions of circles that converge to their common centre, x. With no loss of generality it can be assumed that P does not lie in g_1, where g_1 is the first domain of the chain of domains (Gebietskette) determined by the above chain of cross-cuts. On c, in the order from P to ω, let P_1 be the first point of q_1, and P_1' the last point of q_1; after P_1' let P_2 be the first point of q_2 and P_2' the last point of q_2; after P_2' let P_3 be the first point of q_3 and P_3' the last point of q_3; and so on. Let P_1'' be a point of the arc $P_1'P_2$ (subset of c) which is distinct from P_1' and P_2 and at a distance less than 1 from P_1'. Then P_1 can be joined to P_1'' by an arc P_1P_1'' which lies, except for P_1, wholly in g_1, contains no point of $P_1'P_2$ other than P_1'', and is such that every point of P_1P_1'' is at a distance less than 1 from some point of the arc P_1P_1' (subset of q_1). In general, if P_n'' is a point of the arc $P_n'P_{n+1}$ at a distance from P_n' less than $1/n$, there exists an arc P_nP_n'' joining P_n and P_n'' which lies, except for P_n, wholly in g_n (the n^{th} domain of the chain of domains determined by the chain of cross-cuts $q_1, q_2, q_3, \ldots$), contains no point of $P_n'P_{n+1}$ other than P_n'', and such that every point of P_nP_n'' is at a distance less than $1/n$ from some point of the arc P_nP_n' (subset of q_n). The sum of the arcs PP_1, P_1P_1'', $P_1''P_2$ (subset of $P_1'P_2$), $\ldots$, P_nP_n'', $P_n''P_{n+1}$ (subset of $P_n'P_{n+1}$), $\ldots$, together with the set ω, is an arc t

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*Cf C. Carathéodory, *loc. cit.*, Satz XX.

of prime parts. The conditions of the theorem are satisfied by t and the chain of cross-cuts $q_1, q_2, q_3, \ldots$.

Corollary. Let E_1 *and* E_2 *be distinct prime ends of a simply connected domain,* ω *and* ω' *the sets of principal points of* E_1 *and* E_2*, respectively, and suppose that* ω *and* ω' *are mutually exclusive. Then there exist a simple continuous arc,* t*, of prime parts and chains of cross-cuts defining* E_1 *and* E_2 *such that (i) the extremities of* t *are* ω *and* ω'*, (ii)* $t-\omega-\omega'$ *is a subset of* G*, (iii) every prime part of* t*, except* ω *and* ω'*, is a point, (iv)* $t-\omega-\omega'$ *approximates* ω *in* E_1 *and* ω' *in* E_2*, (v) every cross-cut of the chains defining* E_1 *and* E_2 *has only one point in common with* t.

Definition. A set having the properties of t as stated in this corollary will be called a *cross-cut of prime parts.* If either ω or ω' reduces to a single point, the set will still be called a cross-cut of prime parts. If both ω and ω' reduce to a single point, the set will be called, as usually, a cross-cut.

Theorem F. *If* K *is a cross-cut of prime parts of a simply connected domain* G*, then* K *divides* G *into two mutually exclusive domains,* G_1 *and* G_2.

Proof. Let G be mapped conformally by means of an analytic function $f(z)$ on the interior of the unit circle, $|z|<1$. The image of K, together with its limit points on the circle $|z|=1$, is an arc* which lies, except for its end-points, wholly interior to the unit circle $|z|=1$. This arc divides the interior of the unit circle into two mutually exclusive domains. The remainder of the proof should be obvious.

The following theorem can now be established by means of an argument similar to that used in proving Theorem D:

Theorem D′. *Let* AB *and* CD *be a cross-cut and a cross-cut of prime parts, respectively, of a simply connected domain,* G*, such that the prime ends* E_A *and* E_B *corresponding to* AB *are separated by the prime ends* E_C *and* E_D *corresponding to* CD*. Let the end-points of* AB *be* A *and* B*, and the extremities of* CD *be* ω *and* ω'*, and suppose that* A*,* B*,* ω *and* ω' *are mutually exclusive point-sets. Let* t *be an arc with end-points* A *and* B *and containing no point of* CD*. Under these conditions there exists a simple closed curve* J*, which consists of a sub-arc of* t *and a sub-arc of* AB*, and is such that* ω *is interior to* J *and* ω' *exterior to* J*, or vice versa.*

Theorem G. *Let* E_1 *be a prime end of a simply connected domain* G*,* A *and* B *distinct points contained in* E_1*, and* C *a set irreducibly connected from* A *to* B *such that* $C-A-B$ *is a subset of* G *and* A *and* B *are limit points of* $C-A-B$ *only for the prime end* E_1*. Let* K *be a cross-cut of prime parts of* G*, such that the prime ends corresponding to* K *are* E_1 *and* E_2*;* K *satisfies the conditions stated in the corollary of Theorem E, and neither* A *nor* B *is a principal point of* E_2*. Then if* G_1 *and* G_2 *are the domains into which* K *separates* G*,* A *and* B *are not both limit points of* $G_1\times(C-A-B)$*, or of* $G_2\times(C-A-B)$.

Proof. Suppose both A and B are limit points of $G_2\times(C-A-B)$. Let x and y be distinct points of $C-A-B$, and let the order of x, y, A, and B on C be† A, x, y, B. Let those portions of C from A to x and from y to B

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*Cf. C. Carathéodory, *loc. cit.*, Satz XIII.

†Cf. B. Knaster and C. Kuratowski, *loc. cit.*, Th. XXIII.

be denoted by C_1 and C_2, respectively. Then A is not a limit point of C_2 and B is not a limit point of C_1[*]. But A is a limit point of $C_1 \times G_2$ and B is a limit point of $C_2 \times G_2$.

Let $q_1, q_2, q_3, \ldots,$ be the chain of cross-cuts defining E_1, and composed (see proof of Theorem E) of portions of circles that converge to their common centre, w, a point of ω (the set of principal points of E_1). It can be assumed that for every n, E_2 is not contained in g_n (where $g_1, g_2, g_3, \ldots,$ is the chain of domains determined by the chain of cross-cuts defining E_1). It can also be assumed that q_n does not have A or B as an end-point.

A and B are secondary points (Nebenpunkte) of E_1. For, since A and B are limit points of $C-A-B$ only for the prime end E_1, A and B are limit points of C_1 and C_2, respectively, for the prime end E_1 (Theorem B); and were A, say, a principal point of E_1, then would A be a limit point of C_2[†], which is impossible. As, in addition, neither A nor B is a principal point of E_2, A and B are not points of K.

Hence there exists a circle K_A, with centre A, such that if R_A is the circular domain[‡] bounded by K_A,

$$\bar{R}_A \times (K+q_n+C_2+B)=0; \ (n=1, 2, 3, \ldots),$$

and a circle K_B, with centre B, such that if R_B is the circular domain bounded by K_B,

$$\bar{R}_B \times (K+q_n+C_1+\bar{R}_A)=0; \ (n=1, 2, 3, \ldots).$$

For every positive integer n, that portion of K from P_n, the point common to q_n and K, to ω separates g_n into two mutually exclusive domains, $g_n^{(1)}$ and $g_n^{(2)}$ (Theorem F), which are subsets of G_1 and G_2, respectively. Then A is a limit point of $C_1 \times g_n^{(2)}$ and B a limit point of $C_2 \times g_n^{(2)}$.

Accordingly there exists, common to $g_1^{(2)}$ and R_A, a point A_1 of C_1, and common to R_B and $g_1^{(2)}$ a point B_1 of C_2.

There exists a positive integer, i, such that neither g_i nor q_i contains A_1 or B_1. Common to $g_i^{(2)}$ and R_A there exists a point A_2 of C_1 and common to $g_i^{(2)}$ and R_B there exists a point B_2 of C_2.

Since

$$C_1(\bar{C}_2+\bar{R}_B)=0,$$

A_1 and A_2 lie in a connected subset of G, viz., C_1-A, which contains no point of the closed set $\bar{C}_2+\bar{R}_B$, and hence (as will be demonstrated in Theorem H) A_1 and A_2 are the end-points of an arc, a_1, which lies wholly in G and contains no point of the set $\bar{C}_2+\bar{R}_B$. Similarly, B_1 and B_2 are the end-points of an arc, b_1, which lies wholly in G and contains no points of $\bar{C}_1+\bar{R}_A+a_1$.

On a_1, in the order from A_1 to A_2, let A_1' be the first point of K_A, and A_2' the last point of K_A. Such points will exist since $\bar{R}_A$ and q_i have no points in common. A_1' is a point of G_2, but not of $g_i^{(2)}$, and A_2' is a point of both G_2

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[*] Cf. B. Knaster and C. Kuratowski, *loc. cit*,. Th. XIX.

[†] Cf. C. Carathéodory, *loc. cit.*, § 43.

[‡] *i.e.*, the bounded domain whose boundary is K_A.

and $g_i^{(2)}$ Let H_1 be that maximal connected subset* of $K_A \times G$ determined by A_1', and H_2 that maximal connected subset of $K_A \times G$ determined by A_2'. H_2 and H_1 are subsets of $g_i^{(2)}$ and $G_2 - g_i^{(2)}$, respectively.

There exists a subset, t_1, of $a_1 + \overline{H}_1 + \overline{H}_2$ which has the following properties: (i) t_1 is a cross-cut of G, (ii) if $E_i^{(2)}$ is that prime end of G_2 which corresponds to q_i and is contained in G_2, the prime ends $E_A^{(1)}$ and $E_A^{(2)}$ corresponding to t_1 are separated by E_1 and $E_i^{(2)}$; furthermore, $E_A^{(1)}$ and $E_A^{(2)}$ are prime ends of G contained in G_2, and indeed, $E_A^{(2)}$is contained in $g_i^{(2)}$, but $E_A^{(1)}$ is not contained in $g_i^{(2)}$.

In a similar manner, there can be shown to exist a cross-cut t_2 of G, subset of $a_2 + K_B$, which has the following properties: (i) the prime ends of G corresponding to t_2, $E_B^{(1)}$ and $E_B^{(2)}$, are separated by E_1 and $E_i^{(2)}$; (ii) $E_B^{(1)}$ and $E_B^{(2)}$ are contained in G_2, and $E_B^{(2)}$ is contained in $g_i^{(2)}$; (iii) t_1 and t_2 have no point in common. Property (iii) is a result of the fact that $a_1 + K_A$ and $a_2 + K_B$ have no points in common.

The prime ends E_1, E_2, $E_i^{(2)}$, $E_A^{(1)}$, $E_A^{(2)}$, $E_B^{(1)}$ and $E_B^{(2)}$ must occur in one of the following cyclical orders:

$$(1)\ E_1, E_A^{(2)}, E_B^{(2)}, E_i^{(2)}, E_A^{(1)}, E_B^{(1)}, E_2, E_1,$$

$$(2)\ E_1, E_B^{(2)}, E_A^{(2)}, E_i^{(2)}, E_A^{(1)}, E_B^{(1)}, E_2, E_1,$$

$$(3)\ E_1, E_A^{(2)}, E_B^{(2)}, E_i^{(2)}, E_B^{(1)}, E_A^{(1)}, E_2, E_1,$$

$$(4)\ E_1, E_B^{(2)}, E_A^{(2)}, E_i^{(2)}, E_B^{(1)}, E_A^{(1)}, E_2, E_1.$$

In orders (1) and (4) $E_A^{(1)}$ and $E_A^{(2)}$ are separated by $E_B^{(1)}$ and $E_B^{(2)}$; but t_1 and t_2 have no point in common and hence these orders are impossible.

Consider order (2). Let j be a positive integer such that g_j contains no point of t_2. Let D_1 and D_2 be the domains into which t_2 divides G, D_1 being that one of these domains that contains g_j. As t_1 contains points of H_1, and H_1 contains no points of t_2, H_1 is a subset of D_2. $C_1 - A$ contains a point of H_1, viz., A_1'. Also $C_1 - A$ contains a point, A_3, of g_j. Then $C_1 - A$ is a connected subset of G that contains a point of D_1 and a point of D_2, but no point of t_2. This is impossible. Hence order (2) is impossible.

In a similar manner it can be shown that order (3) is impossible.

Hence the supposition that A and B are both limit points of $G_2 \times (C - A - B)$ leads to a contradiction. In a similar manner it can be shown that the supposition that A and B are both limit points of $G_1 \times (C - A - B)$ leads to a contradiction.

Theorem H. *Let G be a bounded domain, K any closed set of points and N a connected subset of G which contains no points of K. Then every pair of distinct points of N are the end-points of an arc which lies in G and contains no point of K.*

Proof. Let β be the boundary of G; β lies interior to some simple closed curve J. Let I be the interior of J. Then $G + \beta$ is a subset of I†. The set of points $J+I-G$ is a closed set L. The set of points which K has in common with $J+I$ is a closed set M. Then $L+M$ is a closed subset of $J+I$. As N has no points in common with $L+M$, it is clear that N is a connected subset of an open subset, $T[=J+I-(L+M)]$, of $J+I$, and hence every two points of N are the end-points of an arc of T*. As T is a subset of G the theorem is proved.

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*If M is a point-set, and P a point of M, that maximal connected subset of M determined by P is the set of all points $[x]$ of M such that x and P lie in a connected subset of M.

†Cf. R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, 15 (1922), pp. 254-260, Lemma 2.

The following lemma was proved by B. Knaster and C. Kuratowski†:

Lemma 1. *M and N being two mutually separated point-sets, at least one of which is bounded, and A and B being two points of M and N, respectively, there exists a bounded continuum, K, which cuts‡ the plane between A and B and which contains no point of $M+N$.*

Theorem I. *If M and N are two mutually separated connected point-sets, at least one of which is bounded, there exists a simply connected bounded domain, G, which contains M, but does not contain N, or vice versa.*

Proof. If A and B are points of M and N, respectively, then by the above lemma there exists a bounded continuum, K, which cuts the plane between A and B, and hence between M and N. As only one of the domains complementary to a bounded continuum is unbounded, one of the sets M, N, say M, lies in a bounded domain, G, complementary to K. That the boundary of G is connected has been proved by Brouwer§. That G is simply connected follows from a theorem proved by R. L. Moore||.

§3. *Theorem* 3. *In a plane, S, let C_1 and C_2 be point-sets which are irreducibly connected from A to B (points of S), and such that $C_1-(A+B)$ and $C_2-(A+B)$ are mutually separated. Then the set C_1+C_2 cuts the plane S.*

Proof. I. Let one of the sets C_1, C_2, be bounded.

Let

$$C_1-(A+B)=F_1,\ C_2-(A+B)=F_2.$$

That F_1 and F_2 are connected sets has been proved by Knaster and Kuratowski¶.

By Theorem I there exists a simply connected bounded domain, G, which contains, say, F_1, but contains no points of F_2, and whose boundary, β, is a continuum which cuts the plane between F_1 and F_2.

A and B are points of β, since A and B are both limit points of F_1 and F_2, and hence of point-sets interior and exterior to G. Then there exists a prime

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*Cf. R. L. Moore, *Concerning continuous curves in the plane*, *loc. cit.*, Th. 1.

†*loc. cit.*, p. 233, th. 37. Although in the statement of their theorem, Messrs. Knaster and Kuratowski do not mention the boundedness of the continuum K, it clearly follows from their proof.

‡A point-set K *cuts* the plane between two points A and B that do not belong to K if there exists no subcontinuum of $S-K$ (where S is the set of all points in the plane) that contains A and B.

§L. E. J. Brouwer, *Beweis des Jordanschen Kurvensatzes*, Mathematische Annalen 69 (1910), pp. 169-175.

||*Concerning Continuous Curves in the Plane, loc. cit.*, Th. 2.

¶*loc. cit.*, Corollaire XXIV.

end*, E_A, which contains A and is such that A is a limit point of F_1 for this prime end (Theorem B). Similarly, there exists a prime end E_B which contains B, and is such that B is a limit point of F_1 for this prime end.

(a) Suppose E_A and E_B are distinct prime ends. Then they are the ends of two intervals of prime ends, I_1 and I_2, which have in common only E_A and E_B. By Theorem A, there exists in I_1 a prime end E_C which is distinct from E_A and E_B, and which contains a point C that is accessible for E_C and is distinct from A and B. In I_2 there exists a prime end E_D which is distinct from E_A, E_B, and E_C, and which contains a point D that is accessible for E_D and is distinct from A, B, and C. C and D are the end-points of an arc, a, which lies, except for C and D, wholly in G, and which converges to each of the prime ends E_C, E_D.

I shall show that M cuts the plane between C and D. For suppose there exists, in $S-M$, a continuum, K, which contains C and D. Then the set $a+K$ forms a continuum, N. A and B must lie in the same domain complementary to N, since they lie in a connected set, C_2, which contains no point of N.

E_C and E_D are the ends of two intervals of prime ends, J_1, of which E_A is a prime end, and J_2, of which E_B is a prime end. Let q_1, q_2, q_3, . . . , be a chain of cross-cuts defining E_A, such that for any positive integer i there exists a circle with centre at A which neither contains nor encloses any point of q_i. (Theorem C). Let g_1, g_2, g_3, . . . , be the corresponding chain of regions. Then there exists a positive integer, n, such that the prime ends corresponding to q_n are prime ends of J_1 distinct from E_C and E_D, and such that neither g_n nor q_n contains points of a.

Similarly, let t_1, t_2, t_3, . . . , be a chain of cross-cuts defining E_B, such that for every positive integer i there exists a circle with centre at B which neither contains nor encloses any point of t_i. Let u_1, u_2, u_3, . . . , be the corresponding chain of regions. Then there exists a positive integer m such that the prime ends corresponding to t_m are prime ends of J_2 distinct from E_C and E_D, and such that neither t_m nor u_m has points in common with $a+g_n+q_n$.

Let R_A be a circular domain with centre† A and of such a radius that $\bar{R}_A$ contains no point of the set $N+q_n+B$. Let R_B be a circular domain with centre B and of such a radius that $\bar{R}_B$ contains no point of the set $N+t_m+\bar{R}_A$.

Since A is a limit point of F_1 for the prime end E_A, there is a point of F_1, say A_1, which is common to g_n and R_A. Similarly, there is a point of F_1, say B_1, common to u_m and R_B. A_1 and B_1 are distinct points, since R_A and R_B are mutually exclusive. As F_1 is a connected subset of G which contains A_1 and B_1 and contains no point of K, there exists, in G, by virtue of Theorem H, an arc, c, which has A_1 and B_1 as its end-points, and contains no point of K.

Join A and A_1 by a straight line interval, s_1. Let A_2 be the last point of the set $c\times g_n$ on s_1 in the order from A_1 to A. That such a point exists is easily shown as follows:

Let

$$L=\bar{R}_A\times\bar{g}_n\times c.$$

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*Hereafter, unless otherwise indicated, "prime end" will be understood to mean "prime end of G".

†By the *centre* and *radius* of a circular domain R are meant the centre and radius, respectively, of the circle which forms the boundary of R.

L is a closed set, since it is the set of points common to three closed sets. Also, L is non-vacuous, since A_1 is a point of L. Let A_2 be the last point of L on s_1 in the order from A_1 to A. A_2 is not a point of the boundary of g_n, since the only points of this boundary that c contains are points of q_n, and no points of q_n lie in R_A. Hence A_2 is a point of c in g_n. That it is the last point of $c \times g_n$ on s_1 in the order from A_1 to A is obvious.

After A_2, on s_1, in the order from A_1 to A, let A' be the first point of β.

That portion of c from B_1 to A_2, together with that portion of s_1 from A_2 to A', is an arc $A'B_1$, which lies, except for A', wholly in G. Let the prime end which corresponds to this arc be $E_{A'}$. $E_{A'}$ is contained in g_n and is therefore a prime end of the interval J_1, distinct from the ends of J_1.

Join B and B_1 by a straight line interval, s_2. Let B_2 be the last point of the set $A'B_1 \times g_n$ on s_2 in the order from B_1 to B. After B_2, on s_2, in the order from B_1 to B, let B' be the first point of β.

That portion of $A'B_1$ from A' to B_2, together with that portion of s_2 from B_2 to B', is an arc $A'B'$. Besides the prime end $E_{A'}$, there corresponds to this arc a prime end $E_{B'}$, containing B', and belonging to the interval J_2, and distinct from the ends of J_2.

A' and A are in the same domain complementary to N, since R_A contains no points of N. Similarly, B' and B are in the same domain complementary to N. Hence, as A and B lie in the same domain complementary to N, A' and B' lie in the same domain complementary to N.

Hence there exists an arc, t, which has A' and B' as end-points, and which has no points in common with N. *A fortiori*, t and a have no points in common. Hence by Theorem D there exists a simple closed curve, J, which consists entirely of points of t and $A'B'$, and is such that C lies interior to J and D exterior to J, or vice versa. But K has no points in common with t or $A'B'$, and therefore no points in common with J. This is absurd, since K contains both C and D. Thus the supposition that C and D lie in a subcontinuum of $S-M$ must be false and M cuts the plane between C and D.

(*b*) Suppose E_A and E_B are the same prime end E_{AB}. If either A or B is a limit point of F_1 for any prime end distinct from E_{AB}, a method of argument similar to that in (*a*) may be used. If A and B are limit points of F_1 only for the prime end E_{AB}, we may proceed as follows:

Let C be a principal point of E_{AB}. Then C is distinct from A and B, since A and B are secondary points of E_{AB}, as shown in the proof of Theorem G. Let D be a point of β which is not a principal point of E_{AB}, which is distinct from A and B, and which is accessible for a prime end E_D that is distinct from E_{AB}*.

By the corollary of Theorem E there exists a cross-cut of prime parts, a, which has the following properties: (i) ω (the set of principal points of E_{AB}) and D are the extremities of a, (ii) the set $a-\omega-D$ approximates ω in E_{AB} and D in E_D. By Theorem F, a separates G into two mutually exclusive domains G_1 and G_2, and by Theorem G, A and B are not both limit points of one of the sets $F_1\times G_1$, $F_1\times G_2$. Hence A is a limit point of $F_1\times G_1$, say, but not of $F_1\times G_2$, and B is a limit point of $F_1\times G_2$, but not of $F_1\times G_1$.

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*The existence of such a point is easily shown as follows: As E_{AB} contains secondary points, let x be a secondary point of E_{AB} distinct from A and B. Let R be a circular domain with centre x and of such a radius that it contains no principal point of E_{AB}, and neither of the points A, B. R contains some point, y, of G. On a straight line interval joining x and y, let D be the first point of β in the order from y to x.

Suppose that there exists a continuum, K, which contains C and D, but contains no points of M, and let $a+K=N$.

Let K_A be a circular domain with centre A and of such a radius that $\overline{K}_A$ contains no points of the set $N+B+F_1\times G_2$. Let K_B be a circular domain with centre B and of such a radius that $\overline{K}_B$ contains no points of the set $N+\overline{K}_A+F_1\times G_1$. Let A_1 and B_1 be points of F_1 in K_A and K_B, respectively. There exists, in G, by Theorem H, an arc c which has A_1 and B_1 as end-points and contains no point of K.

Join A and A_1 by a straight line segment interval, s_1. Let A_2 be the last point of $c\times G_1$ on s_1 in the order from A_1 to A, and after A_2 let A' be the first point of β. That portion of c from B_1 to A_2, together with that portion of s_1 from A_2 to A' is an arc $A'B_1$, which lies, except for A', wholly in G. The prime end, $E_{A'}$, which corresponds to this arc, is a prime end of G distinct from E_{AB} and E_D. Furthermore, $E_{A'}$ is a prime end of G_1, since A_1 is a point of G_1.

Join B and B' by a straight line s_2. Let B_2 be the last point of $A'B_1\times G_2$ on s_2 in the order from B_2 to B, and after B_2 let B' be the first point of β. That portion of $A'B_1$ from A' to B_2, together with that portion of s_2 from B_2 to B', is an arc $A'B'$. Besides $E_{A'}$, there corresponds to this arc a prime end $E_{B'}$, distinct from E_{AB} and E_D and also a prime end of G_2.

As $E_{A'}$ and $E_{B'}$ are prime ends of G_1 and G_2, respectively, as well as prime ends of G, they are separated by E_{AB} and E_D.

A' and B' lie in the same domain complementary to N, and hence there exists an arc, t, which has A' and B' as end-points and contains no points of N.

(The remainder of the proof is similar to that of (a), except that Theorem D′ is used to obtain the contradiction instead of Theorem D.)

II. If neither of the sets C_1, C_2, is bounded, proceed as follows:

Let x be a point of C_1-A-B, and let R be a circular domain with centre x and containing no points of C_2. Let P be a point of R which does not belong to C_1, and let K_P be a circle with centre P and lying wholly in R. By an inversion of the plane, S, about K_P, C_2 goes into a set C_2', interior to K_P and therefore bounded. If M' denotes the set into which C_1+C_2 goes under this inversion, the proof of I may be used to show that M' cuts the plane between two points C' and D' of $S-M'$, neither of which is identical with P, and hence if the images under the inversion of C' and D' are C and D, respectively, M cuts the plane between C and D.

In selecting the points C' and D' the centre of K_P must be avoided. In case (a) of I this causes no difficulty if Theorem A is applied. In case (b) of I, if E_{AB} is a prime end of the fourth kind, C' may be selected as a principal point of ω distinct from P, the centre of K_P. However, if E_{AB} is of the second kind, P may be the principal point of E_{AB}. Then the proof of I (b) shows that M' cuts the plane between P and some point D'. Let R_P be a circular domain with centre P and of such a radius that R_P contains neither A' nor B', the images, respectively, of A and B. There exist points of β in R_P which are joined to P by continua of β that lie wholly in R_P and hence contain neither A' nor B', and therefore no point of M'*. Let C' be such a point. Then M' cuts the plane between C' and D'.

*Cf. Anna M. Mullikin, *Certain theorems relating to plane connected point-sets*, Trans. Amer. Math. Soc., 24 (1922), pp. 144-162.

CONCERNING LIMITING SETS IN ABSTRACT SPACES

BY

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Reprinted from the
TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY
Vol. 30, No. 4, pp. 668-685

CONCERNING LIMITING SETS IN ABSTRACT SPACES*

BY

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By a *limit point* of a point set M we mean a point which is the *limit*† of an infinite sequence of distinct points belonging to M. This notion may be generalized by regarding a *point set* as being the limit, in some sense, of a *collection* of point sets. We shall give a precise formulation of such a generalization: A point set K is said to be *the limiting set* of a collection G of point sets provided that it is the set of all points P such that there exists a sequence of points $P_1, P_2, P_3, \cdots$, and a sequence $g_1, g_2, g_3, \cdots$, of *distinct* elements of G such that P is the limit of the sequence $P_1, P_2, P_3, \cdots$, and for every n, P_n belongs to g_n.‡ Theorems concerning such limiting sets are frequently

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* Presented to the Society, September 10, 1925, and September 6, 1928; received by the editors in June, 1926, and January, 1928. The author proved Theorems 1, 16, and 17 while holding a National Research Fellowship.

† The theorems in this paper will be stated for a space L Fréchet or for a specialization of such a space. A space L is a class of elements, which we shall call *points*, such that (1) if P is an element of L and $P_1, P_2, P_3, \cdots$ is a countable sequence of elements belonging to L, then the statement *P is the limit of the sequence $P_1, P_2, P_3, \cdots$* has a definite meaning and the question whether this statement is true or false has a determinate answer as soon as the element P and the sequence in question are themselves determined; (2) if the element P is the limit of the sequence $P_1, P_2, P_3, \cdots$, and $n_1, n_2, n_3, \cdots$ is a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$ then P is the limit of $P_{n(1)}, P_{n(2)}, P_{n(3)}, \cdots$ (where, for typographical reasons, $n(i)$ is used in subscripts in place of n_i); (3) if P is an element of L, P is the limit of the sequence $P, P, P, \cdots$ all of whose elements coincide with the point P. Cf. M. Fréchet, (I) *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), pp. 5–6. A space D is a space L in which with every pair of points A and B there is associated a number $d(A, B)$ such that (1) $d(A, B)=d(B, A)\geqq 0$, (2) $d(A, B)=0$ if and only if $A=B$, (3) if A, B, and C are any three elements then $d(A, B)+d(B, C) \geqq d(A, C)$, and (4) the limit of the sequence $P_1, P_2, P_3, \cdots$ is the point P if and only if $d(P_n, P)$ approaches zero as n approaches infinity. Cf. Fréchet, loc. cit., p. 30, and also (II), *Les ensembles abstraits et le calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1910), p. 2, paragraph 3, and (III), *Sur les ensembles abstraits*, Annales Scientifiques de l'École Normale Supérieure, (3), vol. 38 (1921), p. 350, lines 1–10. The symbol $d(A, B)$ is read "the distance from A to B."

‡ Cf. R. L. Moore, *Report on continuous curves from the view-point of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), p. 297. Concerning variations of this definition (mostly for less general spaces), and also of definitions, to be given later, of other types of limiting sets, see the following papers: L. Zoretti, (1), *Sur les fonctions analytiques uniformes qui possèdent un ensemble parfait discontinu de points singuliers*, Journal de Mathématiques Pures et Appliquées, (6), vol. 1 (1905), pp. 8–9, and (II), *Un théorème de la théorie des ensembles*, Bulletin de la Société Mathématique de France, vol. 37 (1909), pp. 116–119; A. Schoenflies, *Bemerkung zu meinem zweiten Beitrag zur Theorie der Punktmengen*, Mathematische Annalen, vol. 65 (1908), pp. 431–432; S. Janis-

applicable in point set theory and in abstract topology; as an example we may mention theorems concerning conditions under which the limiting set is connected (cf. Theorems 11 and 12, and the discussion and references there given).

It is of interest to discover what properties hold for limiting sets in euclidean spaces, and to determine the general classes of abstract spaces in which such theorems hold. One of the purposes of the study of abstract spaces is the discovery of conditions, preferably necessary and sufficient, that a given theorem should hold in that space. As an example we may consider the result shown in Theorem 1 of this paper. In ordinary space the limiting set of every collection of point sets is closed. We show in Theorem 1 that in a space L a necessary and sufficient condition that the limiting set of every infinite collection of point sets be closed is that the space under consideration be a space S.* Thus, the majority of the properties of a euclidean space have little bearing on the fact that in such a space the limiting set of a collection of point sets is closed.

Another example of such a study is the one we make of the distributive property. A space is said to have *the distributive property*† or *to be distributive* provided that if in that space K is a closed point set and G is a collection of point sets and each point of K belongs to some sub-set of K which is the limiting set of some sub-collection of G, then K is itself the limiting set of a sub-collection G_1 of G. If it be specified in addition that G_1 be countable, the space is said to have the *countably distributive property* or to be *countably distributive*. We show in this paper that every distributive space D is locally compact‡ and further that in order that a space D have the countably

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zewski, *Sur les continus irréductibles entre deux points*, Journal de l'École Polytechnique, (2), vol. 16 (1912), pp. 93–94; F. Hausdorff, (I), *Grundzüge der Mengenlehre*, Leipzig, Veit, 1914, pp. 233–239 and 296–304, and (II), *Mengenlehre*, Berlin, de Gruyter, 1927, pp. 145–150; L. Vietoris, *Stetige Mengen*, Monatshefte für Mathematik und Physik, vol. 31 (1921), pp. 184–194; Zoretti-Rosenthal, *Mengen die von einem Parameter abhangen*, Encyklopaedie der Mathematischen Wissenschaften, Band II, 3, Heft 7 (II C 9a), §15, pp. 938–941.

* The *derived set* of a point set is the set of all its limit points. A *space S* is a space L in which the derived set of every point set is closed. Cf. M. Fréchet, (IV), *Relations entre les notions de limite et de distance*, these Transactions, vol. 19 (1918), p. 55. If M is a point set, M' denotes the derived set of M, and $\bar{M}=M+M'$. $M\subset N$ means that *M is a sub-set of N*.

† The term is due to R. L. Moore. The question whether a bounded euclidean space has this property was raised by W. L. Ayres during a discussion of the proof of a special case of my Theorem 13, and was settled in the affirmative by R. L. Moore.

‡ A point set is said to be *compact* provided that each of its infinite sub-sets has at least one limit point. (Cf. Fréchet I, pp. 6–7.) A space D is said to be *locally compact at a point P* or to be *compact near P* provided that P belongs to a compact open set. A point set is said to be an *open set* provided that its complement in the space under consideration is closed. A space is *locally compact* provided that it is locally compact at each of its points. A point set is *separable* provided that it

distributive property it is necessary and sufficient that it be locally compact and separable.

Certain theorems concerning limiting sets have an analogy to covering theorems. The question concerning the possibility of replacing any collection of point sets in a space by a countable sub-collection having the same limiting set is analogous to the one whether the Lindelöf property* holds in the space. This question is, however, of less interest than the following one: Under what conditions is a point set the limiting set of a given collection of point sets? (See Theorems 9, 10, 12, 13, and 14.) The distributive property is of especial interest in this connection; in a space possessing this property we have a condition which may be described roughly as follows: If the property of being the limiting set of a sub-collection of G is distributed everywhere in a closed point set K, then K itself has this property. It was for the reason just indicated that the term "distributive property" suggested itself to Moore.

Return to the consideration of a hyperspace whose elements are point sets in a given space. Condition (2) mentioned in the definition of a space L is not satisfied if we interpret "the limit of a sequence" as the limiting set of a sequence of point sets.† This objection does not apply to the sequential limiting set. The limiting set of an infinite collection of point sets is called *the sequential*‡ *limiting set* of this collection if and only if every infinite sub-collection of the given collection has the same limiting set; a collection of point sets is *convergent* if it has a non-vacuous sequential limiting set and is *divergent* if it has a vacuous sequential limiting set.§ In Theorems 16 and 17 we show that the aggregate of all closed point sets in an abstract space may be regarded as a space L provided that we introduce a suitable definition for "the limit of a sequence," which involves the notion of a sequential limiting set. In this connection Theorems 5, 6, and 7 are of interest; Theorem 6 shows that there exist spaces L and even spaces D in which no collection of point sets contains a convergent or a divergent sub-sequence.

Janiszewski‖ gives a definition of the limit set of a sequence of point sets;

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contains a countable sub-set such that every point of the set either belongs to this sub-set or is a limit point of it. Cf. Fréchet III, p. 341.

* A collection of open sets is said to *cover* a point set K provided that each point of K belongs to some element of the given collection. A space is said to have the *Lindelöf property* provided that for every collection of open sets covering a point set in it there exists a countable sub-collection covering the point set. Lindelöf first showed that a euclidean space has this property; cf. his paper, *Remarques sur un théorème fondamental de la théorie des ensembles*, Acta Mathematica, vol. 29 (1905), p. 188.

† Cf. Janiszewski, loc. cit., pp. 94–95, example 1.

‡ Cf. R. L. Moore, loc. cit., and Hausdorff I, loc. cit.

§ Cf. Hausdorff I, pp. 230 and 232.

‖ Loc. cit., p. 94.

we modify his definition as follows: The *limit set* of a collection of point sets is the set of all points Q such that Q belongs to the limiting set of every infinite sub-collection of the given collection. It follows from our definition that if P is a point in the limiting set of a collection G of point sets, then P belongs to the *limit* set of some infinite sub-collection of G; we show in Theorem 6 that it is not true in general, even in a space D, that P belongs to the *sequential limiting set* of a sub-collection of G.

Janiszewski* shows that the limit set may be a proper sub-set of the limiting set, and may even be vacuous. In fact, the sequential limiting set exists if and only if the limit set is the same as the limiting set; and, in case this is true, the three are identical. The limiting set of an infinite sub-collection of a collection of point sets is a sub-set of the limiting set of the collection. It is not implied in our definitions that all the elements of a collection of point sets should be distinct; it is possible, for instance that each element should consist of a single point, and that all are the same point; in this case the limiting set of the collection would be the given point.

In concluding the introduction the author wishes to thank R. L. Moore, who indicated to one of his classes the desirability of a systematic study of limiting sets, and whose criticisms and suggestions have been of great value in the preparation of this paper.

THEOREM 1. *In order that every collection of point sets in a space L should have a closed limiting set it is necessary and sufficient that the space be a space S.*

If the condition is not sufficient, there must exist a space S in which there exists a collection G of point sets whose limiting set is not closed. Let P be a point which does not belong to K, the limiting set of G, but is the sequential limit point† of a sequence $U=Q_1+Q_2+Q_3+\cdots$ of distinct points belonging to K. Let F_n be the aggregate of all elements g of G such that, for some integer $m>n$, g contains the point Q_m. It follows that $F_{n+1}\subset F_n$. Then for some sufficiently large value N of n, F_N will contain at most a finite number of elements; otherwise, as may easily be seen, P would have to belong to the limiting set of G. Let $F=G-F_N$, $P_i=Q_{N+i}$, and $T=\sum P_i(i=1,2,3,\cdots)$.

It follows from the definition of P_i and the fact that P_i is a point of no element of F that there exists for each positive integer i a sequence

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* Loc. cit.

† A point is the *sequential limit point* of a sequence of points M provided that it is the limit of every infinite sub-sequence of M. Cf. R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 134.

$T_i = \sum P_{ji}(j=1, 2, 3, \cdots)$ of distinct points such that (1) P_i is the sequential limit point of T_i and (2) for each integer i, P_{ji} belongs to an element g_{ji} of F such that if $m \neq n$ then $g_{mi} \neq g_{ni}$.

Let $S_1 = Q_{11} = P_{11}$ and let $h_{11} = G_1$ be a definite element of F containing P_{11}. For $k > 1$ let $G_k = \sum h_{ki}(i=1, 2, 3, \cdots, k)$ be a sequence of elements of F and $S_k = \sum Q_{ki}(i=1, 2, 3, \cdots, k)$ be a sequence of points selected as follows: h_{ki} is an element of $\sum g_{ji}(j=1, 2, 3, \cdots, \infty)$ with $j = n_{ki}$, where n_{ki} is the smallest positive integer x such that g_{xi} is not one of the finite number of elements of F belonging to $G_1, G_2, \cdots, G_{k-1}$; Q_{ki} is the point $P_{n(k,i),i}$.* Let $W = \sum S_i(i=1, 2, 3, \cdots, \infty)$. For every integer i, W contains infinitely many points in common with T_i, and hence T is a sub-set of the derived set of W. Since W is a sub-set of a space S, its derived set is closed. It follows that there exists a set $Q_{k(1),i(1)}, Q_{k(2),i(2)}, Q_{k(3),i(3)}, \cdots$, of distinct points of W having P as a sequential limit point. But no element of G belongs to more than one of the collections G_i, each of which contains only a finite number of elements. Hence the sequence $h_{k(1),i(1)}, h_{k(2),i(2)}, h_{k(3),i(3)}, \cdots$ contains infinitely many distinct elements of G, and P belongs to the limiting set of this sequence and thus to the limiting set of G. Thus we have demonstrated the sufficiency of the condition.

Suppose that there exists a space L in which the limiting set of every collection of point sets is closed, but which is not a space S. Then there exists in this space a point set M whose derived set has a limit point P which does not belong to M'. Let $W = P_1 + P_2 + P_3 + \cdots$ be a sequence of distinct points belonging to M' and having P as a sequential limit point. Then for each positive integer i there exists a sequence $W_i = P_{1i} + P_{2i} + P_{3i} + \cdots$ which has P_i as a sequential limit point, is a sub-set of M, and does not contain P. Let G be a collection whose elements are P_{xy}, where x and y are independent variables, each having as range the set of positive integers. Then W is a sub-set of the limiting set of G, and so is P, since the limiting set is closed. It follows that P is a limit point of M, contrary to its definition. Thus the condition is necessary.

Definition. A point P is said to be a *point of condensation*† of a point set M provided that every open set containing P contains uncountably many points of M.

Theorem 2. *In order that it should be true for each infinite collection of point sets in a space D that at most a countable number of its elements fail to*

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* For typographical reasons when n_i and n_{ij} occur as subscripts or superscripts they will be printed $n(i)$ and $n(i, j)$ respectively.

† Cf. Lindelöf, loc. cit., p. 184, and Fréchet I, p. 6.

be sub-sets of its limiting set, it is necessary and sufficient that the space be separable.

We shall show first that the condition is sufficient. Let K be the limiting set of a collection G of point sets in a separable space D. Suppose that there exists an uncountable sub-collection $\overline{G}$ of the elements of G such that no element of $\overline{G}$ is a sub-set of K. There must exist a positive number y such that there exist uncountably many elements of $\overline{G}$ containing points whose distances* from K are greater than y. Let E be the set of all points belonging to elements of $\overline{G}$ and having distances from K greater than y. If a point of E is common to infinitely many elements of $\overline{G}$ it belongs to the limiting set of G. This is clearly impossible. Further, if E contains at most a countable number of points, it follows from what we have just proved that at most a countable number of elements of $\overline{G}$ contain points in E; this involves a contradiction with our definition of E. Hence E contains uncountably many points, and has a point of condensation.† Since no point of E belongs to more than a finite number of elements of $\overline{G}$, this point of condensation belongs to the limiting set of G. Since its distance from K is not less than y we have a contradiction. Hence $\overline{G}$ does not exist, and the condition given in the statement of the theorem is sufficient.

We shall show next that the condition is necessary. Suppose that it is not, and that there exists a non-separable space D in which every infinite collection of point sets has the property mentioned in the statement of the theorem. Then the space contains an uncountable point set H having no limit point.‡ Let G be a collection of elements g such that each g consists of a single point belonging to H and each point of H belongs to a single element of G. It follows that H' is the limiting set of G. From the fact that all except a countable number of the elements of G are sub-sets of H' it follows that H' is non-vacuous. Thus, the supposition that the space is not separable has led to a contradiction.

DEFINITION. A space is said to have the *Cantor-Bendixson property* provided that every closed point set in it is the sum of a perfect set and a countable set. The fact that this property holds in a euclidean space is, as indicated by Lindelöf, loc. cit., the most important part of a theorem by Cantor* and Bendixson.

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* If P is a point and N is a point set, by $d(P, N) = d(N, P)$, *the distance from P to N*, is meant the lower bound of the set of values $[d(P, Q)]$, where Q is a point of N.

† Cf. W. Gross, *Zur Theorie der Mengen in denen ein Distanzbegriff definiert ist*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, vol. 123 (1914), IIa, pp. 805–806, and Fréchet I, p. 6

‡ Cf. Gross, loc. cit., pp. 805–806.

THEOREM 3. *In order that a space D should have the Cantor-Bendixson property it is necessary and sufficient that the space be separable.*

The sufficiency of the condition follows from the work of Fréchet† and Gross†; the argument given by Fréchet is similar to one given for the euclidean case by Lindelöf (loc. cit., pp. 183, 187). The necessity of the condition follows easily from Gross's work. Suppose that there exists a space D which has the Cantor-Bendixson property but is not separable. Then it contains an uncountable point set M which has no limit point;‡ then M is closed. It follows that M is the sum of a perfect set and a countable set. Since M contains uncountably many points, this perfect set is non-vacuous. Every point of the perfect set is a limit point of this set and of M. Thus we get a contradiction.

THEOREM 4. *In order that every infinite collection of point sets in a space D should contain a countable sub-collection having the same limiting set as the collection itself it is necessary and sufficient that the space be separable.*

It was pointed out by W. L. Ayres that the sufficiency of the condition for a bounded euclidean space is established as a consequence of the fact that such a space has the countably distributive property. The same conclusion holds for any space D that has this property. Not all separable spaces D have this property, however; cf. Theorem 9.

Let G be a collection of point sets in a separable space D, and K be its limiting set. Since the space is separable there exists a countable§ sequence $P_1+P_2+P_3+\cdots$ every element of which is a point of K, and such that every point of K is the sequential limit point of a subsequence of it. For each positive integer n there exists an element g_n of G whose distance from P_n is less than $1/n$ and which is distinct from $g_1, g_2, g_3, \cdots, g_{n-1}$, the elements associated respectively with $P_1, P_2, P_3, \cdots, P_{n-1}$. Then K is the limiting set of $g_1+g_2+g_3+\cdots$.

The necessity of the condition may be established as follows: Let G be a collection of point sets in a space D such that (1) each element of G is a single point of the space and (2) for each point x of the space there exists a countable sequence $g_{1x}, g_{2x}, g_{3x}, \cdots$ of elements of G, each of which is the point x. Clearly the limiting set of G is M, the set of all points in the space. By our hypothesis G contains a countable sub-collection G_1 which has the same limiting set. It follows from our definition of a limiting set that if y is a point of M and is not the limit of an infinite sequence of distinct points belonging to M_1, the set of all points belonging to elements of G_1, it is the limit of a sequence $y, y, y, \cdots$ of points of M_1, each of which is the point y. Then y belongs to M_1. Since M_1 is countable, M is separable.

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* Cf. G. Cantor, *Fondements d'une théorie générale des ensembles*, Acta Mathematica, vol. 2 (1883), pp. 405, 409–414, and I. Bendixson *Quelques théorèmes de la théorie des ensembles de points*, Acta Mathematica, vol. 2 (1883), pp. 415–427.

† Cf. Fréchet I, p. 19, and Gross, loc. cit., pp. 805–806, 815–816.

‡ Cf. Gross, loc. cit., pp. 805–806.

§ Cf. Gross, loc. cit., pp. 805–806, and Fréchet I, p. 27.

THEOREM 5. *If, in a space D, the limiting set of a collection G of point sets is separable, then every infinite sub-collection of G contains a sub-collection that has a sequential limiting set.*

The question as to the truth of this theorem for a bounded euclidean space was raised by R. L. Moore, and in June, 1925, was proved independently by him and by me; he pointed out that my method of proof has an advantage over his in that it applies to abstract spaces. After submitting this paper for publication I learned that T. Ważewski has the following theorem in his thesis:[*] In a compact space D every infinite collection of point sets contains a convergent sub-collection. Also, since this paper was submitted for publication Zarankiewicz[†] and Hausdorff have published analogous results, Zarankiewicz showing that in a euclidean space and Hausdorff that in a separable space D every infinite collection of point sets contains a sub-collection which has a sequential limiting set. The argument given by Hausdorff does not suffice to prove the theorem we have stated, since he uses the fact that the complement of the limiting set is separable; however, if the proof given by Zarankiewicz, which is similar to the one discovered by the author, is modified slightly, it suffices for our purposes.

In 1909 Zoretti[‡] proved the following theorem: If, in a plane, G is a collection of continua[§] (1) whose sum is bounded, (2) whose limit set is non-vacuous, and (3) whose limiting set is a continuum, then G contains an infinite sub-collection which has a closed and connected sequential limiting set. The conditions (2) and (3) are unnecessary, for the existence of a sequence having property (2) follows from (1), and (3) follows from (2) by a theorem of Zoretti's, p. 116, loc. cit. In his process of selecting a sequence which has a sequential limiting set, Zoretti makes no use of the fact* that the elements of his collection are continua, and this process is adequate to prove that every infinite collection of point sets in a plane whose sum is bounded contains a convergent sub-collection.

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[*] *Sur les courbes de Jordan ne renfermant aucune courbe simple fermée de Jordan*, Krakow, 1923, p. 24.

[†] Cf. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), pp. 127–129; and Hausdorff II, pp. 147–148.

[‡] Cf. Zoretti II.

[§] A *continuum* is a closed and connected point set. (Cf. Cantor, loc. cit., p. 406.) A point set is *connected* provided that it is not the sum of two mutually exclusive, non-vacuous point sets, neither of which contains a limit point of the other. Cf. N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), p. 303, and Hausdorff I, p. 244.

Proof of Theorem 5. Let K be a separable point set in a space D and G be a collection of point sets having K as a limiting set. Then K has a countable sub-set H such that if X is a point of K it is either a point of H or a limit point of H. For each point of H consider all the spheres† having this point as a center and having radii whose lengths are rational numbers. Let $R_1, R_2, R_3, \cdots$ be the set of the interiors of these spheres. Now proceed as in the paper of Zarankiewicz (loc. cit., pp. 128–129).

THEOREM 6. *In a space L which contains a point set having the power of the continuum and having a vacuous derived set there exists an infinite collection of point sets which has no convergent or divergent sub-collection.*

The following example shows that there exist spaces D which satisfy the hypothesis of the theorem: Let D_1 be an aggregate of objects having the power of the continuum in which all the requirements of a space D are satisfied, subject to the condition that the distance between two distinct elements of D_1 is unity.

Proof of Theorem 6. Consider a space L in which there exists a point set M having the power of the continuum, but having no limit point. Let X be the set of all real numbers x such that $0<x<1$. Then there exists a correspondence such that to each x there corresponds exactly one point P_x of M and conversely. Let the number x be expanded as an *infinite* decimal in the binary scale. Let G be a collection of point sets, $g_1, g_2, g_3, \cdots$, where the set g_n consists of all points of M for which the corresponding elements in X, when given the binary expansion just mentioned, have as their nth digit unity.

Let $T=g_{n(1)}+g_{n(2)}+g_{n(3)}+\cdots$ be a sub-collection of G such that $n_1<n_2<n_3<\cdots$. Let $z=\sum 2^{-n(2i)}$ $(i=1, 2, 3, \cdots)$, and $H=\sum g_{n(2i)}$ $(i=1, 2, 3, \cdots)$. Then P_z is common to all the elements of H and hence belongs to the limiting set of H. Hence the limiting set of T is non-vacuous, and T

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* This fact is used in proving that the limiting set is connected.

† The terms *sphere*, *interior* and *exterior of a sphere*, etc. are defined precisely as in euclidean geometry. Cf. Fréchet I, p. 21, paragraph 34. A *spherical region* is the interior of a sphere.

is not divergent. It follows from our definitions of M and G that a point of M belongs to the limiting set of an infinite sub-collection of G if and only if the point belongs to infinitely many elements of this sub-collection. Hence P_z does not belong to the limiting set of $T-H$. Hence T does not converge.

THEOREM 7. *If the hypothesis of the continuum*[*] *is correct, then in order that every infinite collection of point sets in a space D should contain a convergent or a divergent sub-collection, it is necessary and sufficient that the space be separable.*

The theorem is a consequence of Gross, pp. 805-806, Fréchet I, p. 27, and Theorem 6, and Hausdorff II, pp. 147–148.

THEOREM 8. *Every space D which has the distributive property is locally compact.*

If the theorem is not true, there exists a space D having the distributive property, and containing a point P near which it is not compact and a countably infinite set of points, K, which has no limit point and does not contain P. Let R_1 be a spherical region containing P such that $\overline{R}_1$ contains no point of K. There exists in R_1 a countable point set M_1 having no limit point and not containing P; within R_1 there is a spherical region R_2 with its center at P, of diameter less than one half, and containing no point of M_1. This process may be continued; in general, let R_n be a spherical region of diameter less than $1/n$, which contains no point of the point sets $M_1, M_2, M_3, \cdots, M_{n-1}$, has its center at P, and lies within the region R_{n-1}; let $M_n=P_{1n}+P_{2n}+P_{3n}+\cdots$ be a countable sequence of points which lies within R_n, does not contain P, and has no limit point. Let $K=P_1+P_2+P_3+\cdots$. For every pair of integers i and j let $g_{ij}=P_{ij}+P_j$. Then P_j is the limiting set of $\sum g_{ij}$ $(i=1, 2, 3, \cdots, \infty)$. From the fact that the space under consideration has the distributive property and that K is closed, since it has no limit point, it follows that K is the limiting set of a sub-collection N of $\sum\sum g_{ij}$ (i and $j=1, 2, 3, 4, \cdots, \infty$). Then there exists an integer m such that R_m contains no points belonging to elements of N. It follows that N is a sub-collection of $\sum\sum g_{ij}$ $(j=1, 2, 3, \cdots, m-1; i=1, 2, 3, \cdots, \infty)$. There exists a spherical region R which contains P_{m+1} but contains no point of $K-P_{m+1}$ or of $\overline{R}_1$. But P_{m+1} does not belong to any point sets in the collection $\sum\sum g_{ij}$ $(j=1, 2, 3, \cdots, m-1; i=1, 2, 3, \cdots, \infty)$. It follows that P_{m+1} does not belong to the limiting set of N. Thus, the supposition that the space is not locally compact has led to a contradiction.

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[*] The *hypothesis of the continuum*, as formulated by Cantor, is the following: No uncountable point set has a power less than that of the continuum. Sierpiński, *Sur l'hypothèse du continu*, Fundamenta Mathematicae, vol. 5 (1924), p. 177–187, gives a discussion of a number of interesting theorems whose truths follow from that of this hypothesis.

THEOREM 9. *In order that a space D should have the countably distributive property it is necessary and sufficient that it be locally compact and separable.*

First we shall show the sufficiency of the condition. Let K be a closed point set in a separable, locally compact space D and G be a collection of point sets such that each point of K belongs to a sub-set of K which is the limiting set of some infinite sub-collection of G. Let M be the limiting set of G and $N=M-K$. The point set K is separable* and hence there exists a countable sub-set $H=P_1+P_2+P_3+\cdots$ of K such that $K\subset\overline{H}$. For each integer i there exists a sub-collection $G_i=g_{1i}+g_{2i}+g_{3i}+\cdots$ of G such that the *limit set* of G_i is a sub-set of K and contains P_i.

From the fact that K is closed and that the space is locally compact it follows that for each point Q of N there exists a spherical region R_Q which is compact and such that $\overline{R}_Q$ contains no point of K. From the fact that the space is separable and hence has the Lindelöf property† it follows that there exists a countable sequence $R_1, R_2, R_3, \cdots$ of these regions covering N. If i and j are integers, the region R_j contains points of at most a finite number of elements of G_i; for otherwise, since $\overline{R}_j$ is compact, it would contain a point of the limiting set of G_i, thus contradicting the fact that the limiting set of G_i is a sub-set of K and $\overline{R}_j$ contains no point of K. Hence, for every integer i the sequence G_i contains an infinite sub-sequence F_i such that no element of F_i contains a point in any of the regions $R_1, R_2, R_3, \cdots, R_i$. Let $F=\sum F_i$ $(i=1, 2, 3, \cdots, \infty)$. We shall show that E, the limiting set of F, is a sub-set of K. If it is not, it must contain a point W of N, since F is a sub-collection of G and E is then a sub-set of M. There exists an integer m such that W belongs to R_m. Let $n\geqq m$. Then it follows from the definition of F_n that F_n contains no point within R_m. Further, as we have pointed out above, if $1\leqq p\leqq m$, at most a finite number of the elements of G_p, and hence of F_p, contain points within R_m. Hence R_m contains points of at most a finite number of elements of F, and hence W cannot belong to the limiting set of F.

We shall show next that K is a sub-set of the limiting set of F. It follows from the definition of F that H is a sub-set of the limiting set of F. If X is a point of $K-H$, it is a limit point of H. By Theorem 1, X is then a point of the limiting set of F. From $E\subset K$ and $K\supset E$ it follows that $K=E$.

Given a space D which has the countably distributive property. If G

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* Cf. Fréchet I, p. 27, and Gross, pp. 805–806.

† Cf. Gross, pp. 805, 806, 809.

is a collection of point sets, it follows by Theorem 1 that its limiting set is closed. By our hypothesis G can be replaced by a countable sub-collection having the same limiting set. By Theorem 4 the space is separable. By Theorem 8 the space is locally compact. Hence, the condition is necessary.

DEFINITIONS. If M and N are two point sets in a space D, by $l(M, N)$, *the lower distance from M to N*, is meant the lower bound of the values $d(P, N)$, where P is a point of M; and by $u(M, N)$, *the upper distance from M to N*, is meant the upper bound of the values $d(P, N)$. If G and H are collections of point sets, we shall say that H is *upper semi-continuous with respect to G* provided that if g is a fixed element of G and h is a variable element of H, then as $l(h, g)$ approaches zero so does $u(h, g)$. If the collection G is upper semi-continuous with respect to itself, it is said to be an *upper semi-continuous* collection.*

THEOREM 10. *If in a space D, K is a closed point set and G is a collection of point sets, then a sufficient condition that K be the limiting set of G is that K is a sub-set of the limiting set of G and* (1) *contains points in common with the limiting set of every infinite sub-collection of G that has a non-vacuous limiting set*; (2) *G is upper semi-continuous with respect to the point set K. If it be specified in addition that the space be compact, the condition mentioned above is both necessary and sufficient.*

It is easy to construct examples showing that the condition mentioned in the theorem does not remain necessary in case the stipulation that the space be compact be removed. The theorem remains true if we replace (1) by (1′) following: (1′) *If e is a positive number, there exist at most a finite number of elements g of G such that $l(g, K) > e$.*

Proof of Theorem 10. Let H be the limiting set of G. If H contains a point x not belonging to K, there must exist an infinite sub-sequence $G_x = g_1 + g_2 + g_3 + \cdots$ of G such that every element of this sequence is at a distance from x less than $\frac{1}{2}d(x, K) > 0$, and for each positive integer i, $d(x, g_i) < 1/i$. By the hypothesis there exist (1) a positive number d such that when $l(g, K) < d$ then $u(g, K) < \frac{1}{2}d(x, K)$, and (2) an infinite sub-sequence of elements of G_x, every element of this sub-sequence being at a lower distance from K less than d. Thus the supposition that the point x exists has led to a contradiction, and $K = H$.

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* For definitions of lower distance, of upper distance, and of upper semi-continuous collections of point sets, cf. R. L. Moore, *Concerning upper semi-continuous collections of continua which do not separate a given continuum*, Proceedings of the National Academy of Sciences, vol. 10 (1924), p. 356, and *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), p. 416.

We shall now show that if the space is compact the condition is necessary. Part (1) of the condition follows immediately. If part (2) does not follow, there must exist a positive number ϵ and a sub-sequence $H=H_1+H_2+H_3+\cdots$ of elements of G such that, for every n, $u(H_n, K)>\epsilon$, but $l(H_n, K)$ approaches zero as n approaches infinity. Since the space is compact it follows that the limiting set of H contains a point x whose distance from K is not less than ϵ. But the result just stated involves a contradiction of our assumption that K is the limiting set of G and hence contains the limiting set of H.

DEFINITIONS. If x and y are a pair of points, a sequence of points $x_1(=x)$, $x_2, x_3, \cdots, x_n(=y)$ is said to be *a chain joining x and y*; a pair of consecutive points, x_j and x_{j+1}, is said to be a *link* of this chain and $d(x_j, x_{j+1})$ is the *length* of this link. If g is a point set let $f(g)$ be the lower bound of the set of all positive numbers h such that any pair of points in g can be joined by a chain every point of which belongs to g and every link of which has a length less than or equal to h. If g is a bounded point set, $f(g)$ certainly exists; otherwise it need not. If G is *zusammenhängend** in the sense of Cantor, $f(g)=0$.

Let M be a point set in a space L and P and Q be points of M. Let $C_{PQ}=\sum\sum P_{ij}$ ($j=1, 2, 3, \cdots, \infty$; $i=1, 2, 3, \cdots, n_j$) be a sub-set of M such that for each positive integral value of j, $P_{1j}=P$, n_j is a positive integer, and $P_{n(j),j}=Q$. The points P and Q are said *to be well chained in M* provided that there exists at least one C_{PQ} such that if $j_1<j_2<j_3<\cdots$ and the sequence $\sum P_{m(j(k)),j(k)}$ ($k=1, 2, 3, \cdots, \infty$), of the set C_{PQ} has a sequential limit point T, then T is a sequential limit point also of every infinite sub-sequence of $\sum P_{m(j(k))+1,j(k)}$ ($k=1, 2, 3, \cdots, \infty$) and of $\sum P_{m(j(k))-1,j(k)}$ ($k=1, 2, 3, \cdots, \infty$), where $1\leqq m_{j(k)}\leqq n_{j(k)}$. If every pair P and Q of points of M is well chained in M, M is said to be *well chained*.

THEOREM 11. *If, in a locally compact space D, $G=[g]$ is a collection of point sets, K is the limiting set of G, and $f(g)$ approaches zero as $l(g, K)$ approaches zero, then in order that K be well chained it is necessary and sufficient that there exist a well chained sub-set K_1 of K which contains points in common with every sub-set of K which is the limiting set of a sub-collection of G.*

Janiszewski considers an analogous question (1) for a bounded number space and (2) in an abstract space which may be characterized as a perfectly compact, connected space in which a distance is defined (cf. Janiszewski, p. 162, loc. cit.); he shows that in such spaces a sufficient condition that the limiting set be *connected* is that the *limit set exist.* Our theorem is more general than his in that we give necessary and sufficient conditions and that our space is more general.

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* Cf. Cantor, loc. cit., p. 406, and *Ueber unendliche, lineare Punktmannigfaltigkeiten*, Mathematische Annalen, vol. 21 (1883), p. 576.

Schoenflies (loc. cit.) gives a less general case than Janiszewski; he assumes that the sequential limiting set exists. See also Zoretti I, pp. 8–9. and Zoretti II. In connection with Zoretti's work see our comments following Theorem 5. See also Vietoris, loc. cit., p. 186, and Hausdorff I, pp. 301–302.

It might be pointed out that in a non-compact space D the limiting set of a collection of connected point sets need not be connected, cf. Hausdorff I, p. 239, last four lines. As an example of a bounded space D where the condition given in Theorem 11 is not sufficient consider the space D_3 given below. Note, however, that this space is not locally compact. Let H_x be the straight line interval in the euclidean plane having as end points $(x, 0)$ and $(x, 1)$. Let A be the point $(0, 0)$ and B the point $(0, 1)$. Let the points of D_3 be the points in the euclidean sense of $A+B+\sum H_{1/i}$ $(i=1, 2, 3, \cdots, \infty)$, and distance and limit be defined as in ordinary space. Let G be a collection of point sets whose elements are the H's in $\sum H_{1/i}$ $(i=1, 2, 3, \cdots, \infty)$. Then G is a collection of connected point sets having as a sequential limiting set the sum of A and B. The set $A+B$ is neither connected nor well chained.

Proof of Theorem 11. Obviously the condition is necessary; we shall prove its sufficiency.

First we shall show that if M is a connected point set it is "zusammenhängend." For, if it is not, there exists a positive number d and a pair of points x and y of M which cannot be joined by a finite chain of points belonging to M, each link in the chain having a length less than d. Let N be the set of all points of M which can be joined to x by a chain such as that just mentioned. Then both N and $M-N$ are non-vacuous and, since M is connected, there exists a point z which belongs to one of the point sets N and $M-N$ and is a limit point of the other. Let E be a spherical region with center at z and having a radius of length $d/3$. It follows that E contains points of both N and $M-N$. Clearly this involves a contradiction. Evidently M is well chained.

We shall show next that if M and N are two non-compact, connected point sets their sum is well chained. Let $m_1, m_2, m_3, \cdots$ and $n_1, n_2, n_3, \cdots$ be two infinite sequences of points belonging to M and N respectively, and neither of which has a limit point. It follows from the preceding paragraph that for each positive integer i there exists a pair of chains of points $m_{1i}(=m_1)$, $m_{2i}, m_{3i}, \cdots, m_{p(i),i}(=m_i)$ and $n_{1i}(=n_1), n_{2i}, n_{3i}, \cdots, n_{q(i),i}$ $(=n_i)$ belonging to M and to N respectively such that each link in each of these chains has a length less than $1/i$. Let C_i be the chain $m_{1i}, m_{2i}, m_{3i}, \cdots, m_{p(i),i}, n_{q(i),i}, \cdots, n_{2i}, n_{1i}$. It follows from the definitions of $C_1, C_2, C_3, \cdots$ that m_1 and n_1 are well chained in M, and since they may be selected at pleasure it follows that $M+N$ is well chained.

Hence, to prove that K is well chained it suffices to prove that either (1) K is connected or (2) every maximal* connected sub-set of K is non-compact. Suppose that neither of these conditions is realized. Then K has a compact maximal connected sub-set T which does not contain all of K. About each point of T there exists a spherical region such that the sum of its points and limit points is compact. The point set T can be covered by a finite† sub-collection of this collection of regions. Let H be the sum of the points belonging to this finite set of regions and B be the boundary‡ of H. Evidently B contains no point of T. Since $\overline{H}$ is the sum of a finite number of compact point sets, it is compact. Let G_2 be a sub-collection of G having a *limit* set containing a point P of T. Suppose that K_2, the *limiting* set of G_2, contains points without $\overline{H}$; let Q be such a point. Let d be a positive number which is less than each of the numbers $d(P, h)$ and $d(Q, H)$.§ It follows from the hypothesis of the theorem and the definition of G_2, that G_2 contains a sub-sequence $g_1, g_2, g_3, \cdots$ such that for each positive integer n, $f(g_n)$, $d(P, g_n)$, and $d(Q, g_n)$ are each smaller than the smaller of $d/2$ and $1/n$. For each n there exists a finite chain $F_n = x_{1n}, x_{2n}, x_{3n}, \cdots, x_{i(n)n}$ of points belonging to g_n such that (1) all except the last point in this chain belong to $\overline{H}$ but the last does not, (2) the distance of x_{1n} from P is less than $1/n$, and (3) each link has a length less than $1/n$. The limiting set F of $\sum F_n$ $(n=1, 2, 3, \cdots, \infty)$ is closed and connected,‖ contains P, and is a sub-set of T, which is, by definition, a maximal connected sub-set of K. The point set $\sum x_{i(n)-1,n}$ $(n=1, 2, 3, \cdots, \infty)$ is compact and has the same limit points as $\sum x_{i(n),n}$ $(n=1, 2, 3, \cdots, \infty)$. These points are common to K_2, T, and B. But B and T have no points in common, and hence $K_2 \subset \overline{H}$.

Further $K_2 \subset T$. For, if e is greater than zero, we can evidently cover T with a set of spherical regions each of which is compact and has a diameter less than e. In terms of these regions we can then define a point set $\overline{H}_e$ which has properties similar to those of $\overline{H}$, and which contains no point whose distance from T is greater than c. As above, $K_2 \subset \overline{H}_e$. Then $K_2 \subset T = \prod \overline{H}_{1/i}$ $(i=1, 2, 3, \cdots, \infty)$.

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* *A maximal sub-set, having a property P*, of a point set M, is a sub-set N of M which has the property P but is not a proper sub-set of any sub-set of M which has this property.

† Cf. Gross, p. 810, and Fréchet I, pp. 22 and 26.

‡ That is, B is the set of all limit points of H that do not belong to H.

§ Let h be the complement of $\overline{H}$.

‖ Cf. Janiszewski, loc. cit., p. 98, lines 1–12.

It follows from the hypothesis of the theorem that K_2, and hence T, contains points of K_1. By an argument analogous to that used in showing that $K_2 \subset T$ we can show that $K_1 \subset T$.

If there exists a point Y of $K-T$, there exists an infinite sub-collection G_Y of G having a limit set containing Y. It follows from the hypothesis of the theorem that G_Y contains an infinite sub-collection having a limit set containing a point of K_1. We have shown above that the limiting set of such a collection is a sub-set of T. Hence the point Y does not exist.

From $T \subset K$ and $K \subset T$ it follows that $K = T$. But this contradicts our supposition that K is not well chained. Thus we have established the truth of our theorem.

THEOREM 12. *In order that a compact point set in a locally compact space D should be the closed and connected limiting set of a collection of well chained point sets, it is necessary and sufficient that it* (1) *be a maximal well chained sub-set of the limiting set of this collection and* (2) *contain points in common with the limiting set of every infinite sub-collection of the given collection which has a non-vacuous limiting set.*

It is a consequence of the definition of well-chainedness that if P is a limit point of a point set M then $M+P$ is well chained. Let K be the limiting set of a collection of point sets and K_1 be a maximal well chained sub-set of K. By Theorem 1, K is closed. Since K_1 is a *maximal* well chained sub-set of K, it is closed. The argument used in proving Theorem 11 shows that the condition is sufficient. Its necessity is obvious.

If we leave off the condition that the space D under consideration be locally compact, the condition given in Theorem 12 is not sufficient. Consider, for example, the space D_3 mentioned in the discussion following the statement of Theorem 11. The point A satisfies the condition but is not the limiting set of any sub-collection of $\sum H_{1/i}$ $(i=1, 2, 3, \cdots, \infty)$. The point just raised applies also to each of the following two theorems.

THEOREM 13. *If $G=[g]$ is a collection of point sets in a separable, locally compact space D, K is the limiting set of G, $f(g)\rightarrow 0$ as $l(g, K)\rightarrow 0$, and K_1 is a maximal well chained sub-set of K, then there exists a countable sub-collection of G whose limiting set is K_1.*

Let P be a point of K_1. There exists a sub-collection G_P of G whose limit set exists and contains P. Suppose that there exists a point Q of K_P, the limiting set of G_P, such that Q is not a point of K_1. By Theorem 11, P and

Q are well chained in K_P. But then K_1 is not a maximal well chained sub-set of K. Thus, the supposition that K_P is not a sub-set of K_1 leads to a contradiction. The truth of the theorem follows by Theorem 9.

THEOREM 14. *If G is a collection of connected point sets in a compact space D and K_1 is a maximal connected sub-set of the limiting set of G, there exists a countable sub-collection of G whose limiting set is K_1.*

DEFINITION. A collection G of point sets is said to be a *separable collection* provided that it contains a countable sub-collection H such that if g is an element of G it is either an element of H or the limiting set of some infinite sub-collection of H.

THEOREM 15. *If G is a collection of closed point sets in a separable space D, a sufficient condition that G be a separable collection is that it contain a sub-collection H such that* (1) *H has the same limiting set as G, and* (2) *is upper semi-continuous with respect to G. In particular, if the collection G just mentioned is upper semi-continuous, it is a separable collection.*

The condition mentioned in the theorem is not necessary. Consider, for instance, the collection G whose elements are the graphs in the euclidean plane of $x=0$ and of $y=nx$, where $n=1, 2, 3, \cdots$.

Let K be the common limiting set of G and H. By Theorem 2, G is the sum of a pair of sub-collections G_1 and G_2, and H of a pair of sub-collections G_3 and G_4 such that G_1 and G_3 contain at most a countable number of elements and every element of G_2 and of G_4 respectively is a sub-set of K. Since the space is separable, K is separable and contains a countable sub-set $T=P_1+P_2+P_3+\cdots$ such that every point of it either belongs to T or is a limit point of T. For each positive integer n there exists a countable sub-collection H_n of H having a limit set containing P_n. Let $G_5=\sum H_n$ $(n=1, 2, 3, \cdots, \infty)$. If g is an element of G_2, it contains a countable sub-set $T_g=Q_{1g}+Q_{2g}+Q_{3g}+\cdots$ such that every point of g either belongs to T_g or is a limit point of it. For every positive integral i the point Q_{ig} is a point or a limit point of a sub-sequence of T; it can easily be shown that Q_{ig} belongs to the *limit* set of a sequence M_{ig} belonging to G_5 such that no element of M_{ig} is at a lower distance from Q_{ig} greater than $1/i$. By the note following Theorem 10, g is the limiting set of $\sum M_{ig}$ $(i=1, 2, 3, \cdots, \infty)$. Hence G_1+G_5 is a sub-collection of G of the type required to make G a separable collection.

THEOREM 16. *The aggregate of all closed point sets in a space L may itself be regarded as a space L provided that "the limit of a sequence of elements" in this hyperspace be defined as the minimal closed set in the given space L containing the non-vacuous sequential limiting set of the given sequence.*

If M is a *closed* point set in a space L, the sequential limiting set of the sequence $M, M, M, M, \cdots$, all of whose elements are the point set M, is clearly M. Thus, the third condition in the definition of a space L is satisfied in our hyperspace. If N is a point set in a space L, there exists* a point set K which is the minimal closed point set in that space containing N. From the results just indicated and the definition of a sequential limiting set it follows that our hyperspace satisfies the other requirements of a space L.

THEOREM 17. *The aggregate of all closed point sets in a space S is a space L provided that "the limit of a sequence" is defined as the non-vacuous sequential limiting set of the sequence in the given space S.*

The theorem follows from Theorems 1 and 16.

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* Cf. Hausdorff II, p. 230.

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CONCERNING LIMITING SETS IN ABSTRACT SPACES, II.

BY

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Reprinted from the
Transactions of the American Mathematical Society
Vol. 43, No. 3, pp. 482-493
May, 1938

CONCERNING LIMITING SETS IN ABSTRACT SPACES, II[*]

BY

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In his first paper on limiting sets[†] the author considered the distributive property in connection with metric spaces. In this paper we consider the property in connection with more general spaces and show that it and weak additional hypotheses imply that every uncountable point set in the space under consideration (1) is α-compact in itself[‡] and (2) is separable. It is well known that in a metric space properties (1), (2), and the following, (3), are equivalent: (3) Every point set has the Lindelöf property. Sierpiński has shown that (2) and (3) are independent in a space S.[§] In consideration of Sierpiński's result an equivalence involving these properties as stated in Theorem 7 is of considerable interest and is used in showing that (2) holds in Hausdorff space.

Above we discussed certain properties that hold "im grossen." With the help of the first countability axiom, or a more general hypothesis concerning

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[*] Presented to the Society, December 27, 1928, August 30, 1929, and September 9, 1931; received by the editors September 8, 1936 and, in revised form, May 14, 1937.

[†] These Transactions, vol. 30 (1928), pp. 668–685. In a topological space the *limiting set* of an aggregate G of sets is the set of all points P of the space such that every neighborhood of P contains points in common with infinitely many distinct elements of G. The elements of G are understood to be sets $h(\alpha, g)$, where α is a number and g is a point set in the space; for the case $\alpha=0$ let $h(\alpha, g)$ be g, and for other values of α let the elements of $h(\alpha, g)$ be α and the points of g. Thus, we may refer to an element $h(\alpha, g)$ of G as a point set if $\alpha=0$. A topological space is said to have the *distributive property* provided that if in that space K is a closed point set, G is a collection of sets, and if each point of K belongs to some subset of K which is the limiting set of a sub-collection of G, then K itself is the limiting set of a sub-collection of G.

For well known definitions and for general information about topological spaces see: M. Fréchet, *Les Espaces Abstraits et leur Théorie Considérée comme Introduction à l'Analyse Générale*, 1928; F. Hausdorff, (I) *Grundzüge der Mengenlehre*, 1914, and (II) *Mengenlehre*, 1927; K. Menger, (I) *Dimensionstheorie*, 1928, and (II) *Kurventheorie*, 1932; R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932; C. Kuratowski, *Topologie*, 1933; W. Sierpiński, (I) *Introduction to General Topology*, 1934, translated by C. C. Krieger; and P. Alexandroff and H. Hopf, *Topologie*, 1935.

[‡] A space or a point set is *α-compact (in itself)* provided that every uncountable point set in it has a limit point (contains a limit point of itself). Cf. W. Gross, *Zur Theorie der Mengen in denen ein Distanzbegriff definiert ist*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, part IIa, vol. 123 (1914), p. 805.

[§] Cf. W. Sierpiński, (II) *Sur l'équivalence de trois propriétés des ensembles abstraits*, Fundamenta Mathematicae, vol. 2 (1921), pp. 179–188; C. Kuratowski and W. Sierpiński, *Le théorème de Borel-Lebesgue dans la théorie des ensembles abstraits*, Fundamenta Mathematicae, vol. 2 (1921), pp. 172–178.

monotonic families of neighborhoods* the distributive property gives also local compactness and regularity. The author has been unable to prove that the Lindelöf property is among these necessary conditions; if it is, it is possible to state simple necessary and sufficient conditions for the distributive property (see Theorems 14 and 16).

Lemma I. *In a Fréchet space H in order that every point set be separable, it is necessary and sufficient that* (1) *every closed set be separable and* (2) *if a point is a limit point of a point set, it is a limit point of a countable subset of the point set.*

Lemma II. *In a Fréchet space V every monotonic family of neighborhoods of a point contains a sub-collection which is a well-ordered monotonic descending family of neighborhoods of the point.*

Theorem 1. *A space satisfies the first countability axiom if each point in it has a monotonic family of neighborhoods and one of the following holds*: (A) *The space is a Hausdorff space in which every point set is α-compact in itself*; (B) *the space is a space H in which a point is a limit point of a point set if and only if it is a limit point of a countable subset of the point set.*

Proof. Consider first case (A). Let S be the set of all points in the space, P be a point in it, H be a well-ordered monotonic descending family of neighborhoods of P, and K be a well-ordering of the points of $S-P$. Let U_1 be an element of H, P_1 be the first point of K in U_1, and V_1 the first element in H which is a subset of U_1 and of which P_1 is not a point or a boundary point. Suppose that P is not an isolated point of the space. Suppose that U_x, P_x, and V_x have been defined for each ordinal x less than a definite ordinal α. Provided that there exist elements of H common to all V_x's for $x<\alpha$ we shall define U_α, P_α, and V_α as follows: U_α is the first element of H common to all the V_x's for $x<\alpha$; P_α is the first element of K in U_α; V_α is the first element of H which is a subset of U_α and of which P_α is not a point or limit point. Let G be the well-ordered sequence $(V_1, V_2, V_3, \cdots, V_\omega, \cdots, V_\alpha, \cdots)$, where α is such that U_α, P_α, and V_α exist; and let $E=(P_1, P_2, P_3, \cdots, P_\omega, \cdots, P_\alpha, \cdots)$. Clearly each point of E is an isolated point of E. It follows from our condition that E, and hence G, has each a finite or a countable number of elements.

We shall show that each element of H contains an element of G. Suppose this were not true; then, since H is monotonic, it would contain an element U

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* A *complete* family of neighborhoods of a point is one that defines the operation of derivation at that point; cf. Fréchet, pp. 172–173. Such a family is *monotonic* provided that every pair of its elements has the property that one is a subset of the other, and said to be *monotonic descending* with reference to a definite ordering provided that if one element precedes another, the first contains the second. In connection with a space H Fréchet we shall consider the term neighborhood as equivalent to the term "open set"; cf. Fréchet, pp. 186–187.

which is a subset of all elements of G. There exists a first ordinal λ which is greater than all ordinals x such that there exists an element V_x of G. Since H is a well-ordered sequence, there exists a first one of its elements that is common to all elements of G, and this element is by definition U_λ. Then the first point of K in U_λ is P_λ. There exist open sets R_1 and R_2 containing P_λ and P respectively and having no common points. Then P is not a point or limit point of $S-U_\lambda$ or of $S-R_2$; hence, there exists a first element of H that is a subset of U_λ and does not have P_λ on its boundary. This element is by definition V_λ. But, this is contary to the definition of λ. Thus, every element of H contains an element of G; the converse is true. Since H is a family of neighborhoods of P, so is G.†

Consider next case (B). Let P be a limit point of the set of distinct points, $E=(P_1, P_2, P_3, \cdots)$, none of which is P, and let $H=[W]$ be a family of neighborhoods of P. For each n let W_n be an element of H containing no point of $P_1+P_2+P_3+\cdots+P_n$, and $G=(W_1, W_2, W_3, \cdots)$. Then each element of H contains an element of G; for, if some element U of H did not contain an element of G, it would be a subset of every element of G. Hence, if n is any integer, U is a subset of W_{n+1} and does not contain P_n. But this involves a contradiction, since P is a limit point of E. Thus every element of H contains an element of G, and conversely. It follows that G is a family of neighborhoods of P.†

THEOREM 2. *A locally compact Hausdorff space which has the Lindelöf property satisfies the first countability axiom.*

Proof. Let P be a point of our space T. For each point Q of $T-P$ let U_Q and V_Q be mutually exclusive open sets containing Q and P respectively. Then $T-P$ may be covered by a countable sequence $(U_{Q_1}, U_{Q_2}, U_{Q_3}, \cdots)$ of the elements of $[U_Q]$. Let R be an open set containing P such that $\overline{R}$ is compact; let $W_n=R\cdot(T-\sum_{i=1}^{i=n}\overline{U}_{Q_i})$; and let $F=(W_1, W_2, W_3, \cdots)$. Then F is a monotonic descending family of neighborhoods of P.

For, let M be a point set having points distinct from P in every element of F. It may be shown that M has a subset $N=(P_1, P_2, P_3, \cdots)$ of distinct points such that for $P_n \epsilon W_n$ each n. Since $R\supset W_n$, $\overline{W}_n$ is compact; hence N has a limit point X which is a point of $\overline{W}_1\cdot\overline{W}_2\cdot W_3\cdot\cdots$. If X were a point of $T-P$, there would be an integer n such that X belongs to U_{Q_n}. Since $\overline{W}_n$ contains no point of U_{Q_n} we are involved in a contradiction.

Conversely if P is a limit point of a point set K, every element of F contains a point of K distinct from P.

LEMMA III. *In a space H the limiting set of a collection of point sets is closed.*

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† Cf. Fréchet, p. 173.

THEOREM 3. *Every regular space H which satisfies the first countability axiom and has the distributive property is locally compact.*

This theorem may be proved by methods analogous to those used in the proof of Theorem 8 of the author's first paper, p. 677.

THEOREM 4. *Every space V which has the distributive property is α-compact.*

Proof. Suppose there exists a space which satisfies the hypothesis, but contains an uncountable point set M whose derived set is vacuous. Let K be a countable subset of M, $N=M-K$, and $P_1, P_2, P_3, \cdots$ be points of K. For each point x of N and each positive integer n let $g_{xn}=x+P_n$. Let G^* be the aggregate $[g_{xn}]$. For each point x of N there exists a neighborhood R_x of x which contains no point of $M-x$. Hence x is the limiting set of the aggregate $G_x^*=(g_{x1}, g_{x2}, g_{x3}, \cdots)$. Since N has no limit point, it is closed. It follows from our hypothesis that G^* contains a sub-collection G whose limiting set is N. Let $G_x=G\cdot G_x^*$. Since R_x contains no point of any element of $G-G_x$, G_x contains infinitely many distinct elements. Since N is an uncountable point set, and an element of G_y contains the point z of N only if $y=z$, G has uncountably many elements. Hence there exists an integer m such that P_m is common to infinitely many elements of G. This involves a contradiction with the fact that N is the limiting set of G.

THEOREM 4A. *In order that a metric space should have the distributive property, it is necessary and sufficient that it be locally compact and separable.*

This theorem is a consequence of Theorem 4, Gross, loc. cit., pp. 805–806, and Theorems 8 and 9 of the authors first paper, pp. 677–678.

A space S_1 is said to be a *sub-space* of a space S_2 provided that (1) every point of S_1 is a point of S_2, and (2) if P is an arbitrary point and M an arbitrary point set in S_1 then P is a limit point of M in S_1 if and only if it is a limit point of M in S_2.

LEMMA IV. *Every subspace of a space S is a space S, and every subspace of a space H is a space H.*

THEOREM 5. *If a space S has the distributive property, then every regular, locally compact subspace of it has this property.*

Proof. Let S_1 be a space S having the distributive property, and let T be a regular, locally compact sub-space of it. In T let K be a closed point set and G be a collection of sets such that each point P of K belongs to a subset K_P of K which is the limiting set of a sub-collction G_P of G. Let L_P be the limiting set of G_P with respect to the space S_1. Then $K_P=L_P\cdot M$, where M is the set of all points belonging to T. Let N be the sum of all point sets L_P, where the range of P is K; N' be the derived set of N with reference to the space S_1; and $\overline{N}=N+N'$. It follows from the definition of N that $N\cdot M=K$. Suppose that $N'\cdot M$ contains a point Q which does not belong to K. Since K is a closed point set with respect to T, Q is not a limit point of K in either T or S_1. Hence, there exists in T an open set R_1 containing Q such that if $\overline{R}_{1T}$ denotes the sum of R_1 and its limit points in T, then $K\cdot\overline{R}_{1T}$ is vacuous. Since T is locally compact, there exists in it an open set R_2 containing Q such that $\overline{R}_{2T}$ is compact. Let $R_3=R_1\cdot R_2$. Then R_3 is an open set in T and contains Q. Also $\overline{R}_{1T}\supset\overline{R}_{3T}$. It follows that Q is not a limit point of $M-R_3$ in the space S_1. Hence, there exists in S_1 an open set U which contains Q but contains no point of $M-R_3$. Since Q is a limit point of N, there exists in K a point x such that U contains a point y of L_x. Then U must contain points of infinitely many elements of G_x. Since the elements of G_x are subsets of M, and $U\cdot M$ is a subset of R_3, R_3 contains points of infinitely many elements of G_x. Since $\overline{R}_{3T}$ is compact, it must contain a point W which belongs to the limiting set of G_x; since $\overline{R}_{1T}\supset\overline{R}_{3T}$, the latter contains no point of K. This involves a contradiction with the fact that $K\supset K_x$. Hence the point Q does not exist, and $\overline{N}\cdot M=K$.

For each point x of $\overline{N}-K$ let $h_{1x}, h_{2x}, h_{3x}, \cdots$ be the pairs $(x, 1)$, $(x, 2)$, $(x, 3)$, $\cdots$. Let H be the aggregate $[h_{ix}]$, where the range of i is the set of positive integers, and that of x is the point set $\overline{N}-K$. Since S_1 is a space S, $\overline{N}$ is closed, and for every point P of K the following holds: $\overline{N}\supset N\supset L_P$. It follows that each point of $\overline{N}$ belongs to a subset of $\overline{N}$ which is the limiting set in S_1 of a sub-collection of $G+H$. Since S_1 has the distributive property, $G+H$ has a sub-collection G_1+H_1 such that $\overline{N}$ is the limiting set of G_1+H_1 in S_1 and such that G_1 and H_1 respectively are sub-collections of G and H respectively.

Suppose that K is not a subset of K_1, where K_1 is the limiting set of G_1 in S_1. Then K must contain a point E which belongs to the limiting set of H_1. Let R_E be an open set in T containing E such that $\overline{R}_{ET}$ is compact. It follows from an analogous situation above that in S_1 the point E is not a limit point of $M-R_E$ and that there exists in S_1 an open set U_E which contains no point of $M-R_E$. Then U_E contains a point X of an element of H_1. Let B denote the limiting set of G in S_1. Since $B\supset N$, and B is closed, $B\supset\overline{N}$. Since X is a point of $\overline{N}-K=\overline{N}-\overline{N}\cdot M$, and all elements of G are sub-sets of M, X is the unique limit point in S_1 of an infinite subset of M, and U_E, which contains X, contains such a set E_X. Then $R_E\supset M\cdot U_E\supset E_X$. Since X is not a point of T, E_X has no limit point in T. This involves a contradiction with the fact that $\overline{R}_{ET}$ is a compact subset of T. Hence $K_1\supset K$. Since $M\supset K$, it follows that $M\cdot K_1\supset K$.

Since $\overline{N}\supset K_1$, $K=\overline{N}\cdot M\supset K_1\cdot M$. From $K\supset K_1\cdot M$ and $K_1\cdot M\supset K$, it follows that $K=K_1\cdot M$. Hence K is the limiting set of G_1 in T. Thus we have shown that T has the distributive property.

A space H is said to be *nearly a space L* provided that if in that space P is any point, M is any point set, and P is a limit point of M, then P is the derived set of a subset of M.

THEOREM 5A. *Every regular, locally compact subspace of a space H which has the distributive property and is nearly a space L has the distributive property.*

The proof is the same as that for Theorem 5.

THEOREM 6. *In a space H which has the distributive property every point set is α-compact in itself.*

Proof. Suppose the theorem is not true and that S_1 is a space H which has the distributive property but contains an uncountable point set M which contains no limit point of itself. Let T be the subspace of S_1 whose points are the points of M. To show that T has the distributive property adopt the notation of the proof of Theorem 5 and follow this proof to the place where the existence of the collections G_1 and H_1 is established, and suppose as there that K contains a point E not belonging to the limiting set of G_1. Define M' as the derived set of M in S_1. Since the points of elements of H_1 are points of $\overline{N}-K$, it follows that E is a limit point of $M'-M\cdot M'$ in S_1. Since S_1 is a space H, derived sets in it are closed, and E is a point of M' and hence a limit point of M. Since E is a limit point of M, we are involved in a contradiction. Thus E does not exist, and the argument of Theorem 5 shows that T has the distributive property.

By Theorem 4 the set M must have a limit point in T. But this again is contrary to the definition of M. Thus the supposition that the theorem is not true leads to a contradiction.

Note. When the space of our hypothesis is a space S the proof may be simplified. Let S_1 be our space and define M and T as above; then T is a regular, locally compact subspace of S_1, since all its points are isolated. By Theorem 5, T has the distributive property, and by Theorem 4, we are involved in a contradiction.

THEOREM 7. *In order that in a space H each point set either be condensed in itself or be separable, it is necessary and sufficient that every point set be α-compact in itself.*†

Proof. Obviously the condition is necessary. Suppose that it is not sufficient, and that the space contains a point set E which neither contains a condensation point of itself nor is separable. Then, for each point P of E there exists a countable subset $D(P)$ of E such that P is not a point or a limit point of $E-D(P)$. Let T be a well-ordered sequence $(p_1, p_2, p_3, \cdots, p_\omega, p_{\omega+1}, \cdots, p_\alpha, \cdots)$ of the points of E. We shall now define a sequence of the type δ, where δ is the smallest transfinite ordinal of the third class; $U=(q_1, q_2, q_3, \cdots q_\omega, q_{\omega+1}, \cdots, q_\beta, \cdots)$, where $\beta<\delta$.

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† In part our proof of Theorem 7 follows methods used by Sierpiński; cf. Sierpiński (II). See also the introduction for a discussion of the relation of Theorem 7 to some results by Sierpiński and Kuratowski.

Proceed as follows: Let $q_1=p_1$. Let β be a definite ordinal less than δ. Suppose that q_x has been defined for all ordinals $x<\beta$, and let U_β be the set of all q_x's for such x's. Let S_β be the sum of all point sets $D(q_y)$, where q_y is an element of U_β. Let q_β be the first point of T which is not a point or a limit point of S_β.

We shall now show that q_β exists for every ordinal β less than δ. For, if q_β does not exist for all such ordinals β, there must be a first such ordinal λ for which it does not exist. Then it follows from our definitions that each point of E is either a point or a limit point of S_λ. But S_λ is the sum of all point sets $D(q_z)$, where q_z ranges over U_λ; thus S_λ is the sum of a countable number of countable sets and is countable; then E is separable. Thus, the supposition that λ exists leads to a contradiction.

Next we shall show that q_β is an isolated point of U. By definition q_β is not a point or a limit point of U_β. Further, $S_{\beta+1}$, which contains $D(q_\beta)$, contains no point of $U-U_{\beta+1}$. Since $U-q_\beta=U_\beta+(U-U_{\beta+1})$, it follows that q_β is an isolated point of U.

Thus, every point of the uncountable sequence U is an isolated point of U. By our hypothesis, however, U must contain a limit point of itself. Thus, the supposition that our condition is not sufficient has led to a contradiction.

THEOREM 8. *In order that for each infinite collection of point sets in a space H it be true that at most a countable number of its elements fail to be subsets of its limiting set, it is necessary and sufficient that every point set in the space be α-compact in itself.*

Proof. We shall first show that the condition is sufficient. Suppose that it is not and that there exists in our space a point set K and a collection G of point sets such that K is the limiting set of G, but that G contains an uncountable sub-collection G_1, none of whose elements are subsets of K. For each element g of G_1 let P_g be a point which does not belong to K. It follows by our hypothesis that the set $[P_g]$ *contains* a point Q, every neighborhood of which contains infinitely many elements of $[P_g]$. Then Q belongs to the limiting set of G, and we are involved in a contradiction.

Conversely, let M be an uncountable point set in a space which satisfies our condition. Let G_2 be a collection of point sets whose elements are the points of M, no two elements being the same point. Then M' is the limiting set of G, and M' and M have uncountably many points in common. Thus, the condition is necessary.

Theorems 8 and 9 are generalizations of Theorems 2 and 4, respectively, of our first paper, and are of interest in connection with Theorem 7, and also with Theorems 6 and 10A, in that they indicate consequences of the distributive property.

THEOREM 9. *In order that for a space H every infinite collection of sets should contain a countable sub-collection having the same limiting set as the collection itself, it is necessary and sufficient that every point set in the space be separable.*

Proof. To prove the necessity of the condition proceed as follows: Let N be a point set and $M=\overline{N}$. Let G be a collection such that for each point x of N there exists a collection of elements of G, $g_{1x}, g_{2x}, g_{3x}, \cdots$, where g_{nx} is the pair (n, x). Now proceed by methods analogous to those used in the proof of Theorem 4 of the author's first paper.

Consider next the sufficiency. Let G be a collection of sets and K be the limiting set of G. By our condition K contains a countable subset $N=P_1+P_2+P_3+\cdots$ such that $\overline{N}=K$. Then *for each n the point P_n belongs to the limiting set of some countable sub-collection of G.* Suppose that for some definite $P_j=Q$ this is not true. Let g_0 be a definite element of G. For n greater than zero suppose that g_k has been defined for $k<n$. Let $G_n=G-(g_0+g_1+g_2+\cdots+g_{n-1})$; let H_n be the sum of all elements of G_n; and let $F_n=Q_1+Q_2+Q_3+\cdots$ be a countable set such that $\overline{F}_n \supset H_n \supset F_n$. Then Q is a point of F_n'.

For, let R be an open set containing Q. Since Q belongs to the limiting set of G, and hence of G_n, R contains points of infinitely many elements of G_n. If Q were common to infinitely many elements of G_n, it would be common to a countable infinity of such elements and would belong to the limiting set of this countable collection; this, however, is contrary to the definition of Q. Thus R contains points of H_n distinct from Q, and hence points of F_n. Thus $Q \epsilon F_n'$. For each positive integer k let t_k denote a definite element of G_n that contains Q_k. Let T denote the aggregate $(t_1, t_2, t_3, \cdots)$. Since T has a finite or a countable number of elements, Q does not belong to its limiting set. Hence, there must exist an open set U containing Q which contains points of at most a finite number of the elements of T, say of $t_{k_1}, t_{k_2}, \cdots, t_{k_j}$. Then $U\cdot F_n=\sum_{i=1}^{i=j} U\cdot F_n\cdot t_{k_i}$. Since Q belongs to the derived set of $U\cdot F_n$, it follows that for some i, $0<i<j+1$, Q is a limit point of $U\cdot F_n\cdot t_{k_i}$, that is of t_{k_i}. Let g_n be defined as the sum of n and such a t_{k_i}, and $E=(g_1, g_2, g_3, \cdots)$. Then Q is a limit point of each element of E, and hence belongs to the limiting set of E.

Thus, each point P_n of N is a point of the limiting set of some countable sub-collection M_n of G. Let $M=\sum_{i=1}^{i=\infty} M_n$, and let L be the limiting set of M. Then $K=\overline{N}\supset L\supset N$. Since in a space H derived sets are closed, it follows that $L=\overline{N}=K$. But, M is a countable sub-collection of G.

THEOREM 10. *In a Hausdorff space having the distributive property every closed point set is separable.*

Proof. Suppose that a space S_1 satisfies the hypothesis but not the conclusion of our theorem. Then there exists in it an uncountable, closed, non-separable point set E. By Theorems 6 and 7 the set E contains a point of condensation of itself Q; similarly $E-Q$ contains a point of condensation of itself P. Let U and V be mutually exclusive open sets containing Q and P, respectively. Let $M=U\cdot E$, $N=V\cdot E$, and $H=[S_1-(U+V)]\cdot E$. Then one of the three point sets M, N, or H is non-separable; for otherwise E, their sum, would be separable. Consider the two cases: (I) Either M or N is not separable; (II) both M and N are separable. Consider first case (I) and suppose that it is N that is not separable. Let $K=E-\overline{M}$; then $H+N=E-M\supset K\supset N$, and K is not separable. For, suppose that K is separable and has a countable subset K_1 such that $\overline{K}_1\supset K$. By definition H is the product of the two closed sets E and $S_1-(U+V)$, and thus is closed. It follows that every point of $N=K-K\cdot H$ is either a point or a limit point of $N\cdot K_1$. Thus, the supposition that K is separable, involves a contradiction with the hypothesis that N is not separable.

We shall now define certain sequences by an induction process. Let z_0 be a point of $M-Q$, $R_0=z_0$, and $U_0=U$. Now suppose that U_k, R_k, and z_k have been defined for all non-negative integers k less than the definite integer n. Let U_n be an open set containing Q such that $U_{n-1}-\overline{R}_{n-1}\cdot U_{n-1}\supset U_n$, let z_n be a point of $M\cdot(U_n-Q)$, and let R_n be an open set containing z_n such that $\overline{U}_n-Q\supset\overline{R}_n$. Let $F=z_1+z_2+z_3+\cdots$. The existence of U_n, z_n, and R_n for every positive integer n may be shown by making use, in particular, of Hausdorff's Axiom D. Since the open sets $R_1, R_2, R_3, \cdots$ are mutually exclusive, it follows that $F\cdot F'$ is vacuous.

For each point t of $\overline{K}$ and each positive integer n let $g_{tn}=t+z_n$, and G be the aggregate of all such g_{tn}'s. Since $z_1, z_2, z_3, \cdots$ are distinct points, the limiting set of the aggregate $(g_{t1}, g_{t2}, g_{t3}, \cdots)$ is $t+F'$. Thus, each point of $\overline{K}+F'$ belongs to a subset of $\overline{K}+F'$ which is the limiting set of a sub-collection of G. Hence, G has a sub-collection G_1 whose limiting set is $\overline{K}+F'$. Let W be the product of $\overline{K}$ and the sum of the elements of G_1. Since $E-\overline{M}=K$, and $M\supset F$, it follows that $K\cdot\overline{F}$ is vacuous. Then every point of K is a point or a limit point of W, and so is every point of $\overline{K}$. If the elements of G_1 were countable, so would be the points of W; then $\overline{K}$ would be separable. This is impossible, since K is not separable, and no point of K is a limit point of $\overline{K}-K$. Hence, there are uncountably many elements of G_1, and there must exist an integer j such that z_j belongs to uncountably many elements of G_1. Then z_j belongs to the limiting set of G_1, that is to $\overline{K}+F'$. But, z_j belongs to neither $\overline{K}$ nor F'. Thus, case (I) involves a contradiction.

Consider next case (II). Since both M and N are separable, so are $\overline{M}$ and $\overline{N}$. Let $K=E-(\overline{M}+\overline{N})$. Then K is not separable. Define first F and then G precisely as in the proof of case (I); by following this proof we again arrive at a contradiction. Thus, the supposition that the theorem is not true is untenable.

THEOREM 10A. *If a Hausdorff space is nearly a space L and has the distributive property, every point set in it is separable.*

This is a consequence of Theorems 10 and 6 and Lemma I.

THEOREM 11. *A space H which satisfies the first countability axiom is a Hausdorff space if and only if it is a space S.*[†]

THEOREM 12. *A locally compact space S (Hausdorff space) which satisfies the first countability axiom is regular.*[‡]

THEOREM 12A. *If a Hausdorff space is locally compact at one of its points P, and P has a countable family of neighborhoods, the space is regular at P.*

THEOREM 13. *A space S (Hausdorff space) which has the distributive property and satisfies the first countability axiom is regular.*

Proof. Suppose that S_1 is a space S which satisfies the hypothesis of the theorem but contains a point P at which it is not regular. Then there exists in S_1 an open set R containing P such that if R_i is any open set whatever containing P, then $\overline{R}_i$ is not a subset of R. Let $(V_1, V_2, V_3, \cdots)$ be a countable family of neighborhoods containing P such that $R \supset V_1 \supset V_2 \supset V_3 \supset \cdots$. Let $U_1=V_1$, $m_1=1$, and M_1 be a countable subset of V_1-P which has a unique limit point in $R'-R$, say P_1. By an induction process we shall now define U_n, M_n, m_n, and P_n for every positive integer n. Proceed as follows: Suppose they have been defined for all n's less than a definite integer k. Let m_k be the first integer greater than m_{k-1} such that V_{m_k} contains no point of $\sum_{j=1}^{j=k-1} M_j$, let $U_k=V_{m_k}$, and let M_k be a compact countable sequence of points belonging to U_k-P and having a unique limit point P_k belonging to $R'-R$.

[†] Cf. Hausdorff (1), pp. 263–265, and Fréchet, *Démonstration de quelques propriétés des ensembles abstraits*, American Journal of Mathematics, vol. 50 (1928), p. 65.

[‡] Cf. Alexandroff and Urysohn, *Mémoire sur les espaces topologique compacts*, Verhandelingen, Koninklÿjke Akademie van Wetenschappen, Amsterdam, vol. 14 (1929), pp. 28–29.

We shall now prove that U_k, M_k, m_k, and P_k exist for all positive integers k; the argument suggests, in particular, the proof for the case $k=1$. Suppose that our proposition has been established for all n's less than k, where $1<k$. Then each of the sequences $M_1, M_2, M_3, \cdots, M_{k-1}$ has exactly one limit point; the derived set of their sum is $P_1+P_2+\cdots+P_{k-1}$, which is a subset of $R'-R$. Then P is not a limit point of the closed point set $\sum_{j=1}^{j=k-1} M_j$, and there exists an integer m_k greater than m_{k-1} such that V_{m_k} contains no point of this point set. Hence U_k exists. It follows from the definition of R that $\bar{U}_k$ is not a subset of R, and that U_k has a limit point P_k in $R'-R$. Since our space is a space S, U_k-P has a countable subset M_k such that P_k is the unique limit point of every infinite subset of M_k. Our existence theorem may thus be established by mathematical induction.

The sequence $P_1, P_2, P_3, \cdots$ contains a sub-sequence $P_{n_1}, P_{n_2}, P_{n_3}, \cdots$, having not more than one limit point, such that $n_1<n_2<n_3<\cdots$. If this sub-sequence has a limiting set, let it be denoted by the symbol Q; otherwise, let Q be the null set. Let $P_{n_k}=Q_k$; $M_{n_k}=N_k$; let $O_{1k}, O_{2k}, O_{3k}, \cdots$ be the points of N_k; $g_{jk}=Q+O_{k1}+O_{jk}$; let G_k be the sequence $(g_{1k}, g_{2k}, g_{3k}, \cdots)$; $G^*=\sum_{j=1}^{j=\infty} G_j$; and $K=Q+N_1+\sum_{k=1}^{k=\infty} Q_k$. The limiting set of G_k is $Q+O_{k1}+Q_k$. Since K is closed and the space has the distributive property, G^* contains a sub-collection G whose limiting set is K. Suppose that G contains elements in common with at most a finite number of the elements of the aggregates $G_1, G_2, G_3, \cdots$, say with those having subscripts not greater than a definite integer t. Then, contrary to the fact that K contains infinitely many distinct points, the limiting set of G is a subset of $Q+\sum_{k=1}^{k=t}(O_{k1}+Q_k)$. Hence, for infinitely many values of k there exist elements of G which contain points of N_k. Thus, every element of $(V_1, V_2, V_3, \cdots)$ contains points in common with infinitely many distinct elements of G, and P belongs to the limiting set of G. Thus, the supposition that our space is not regular has led to a contradiction.

THEOREM 13A. *If a space S has the distributive property and a point P in it has a countable family of neighborhoods, the space is regular at P.*

This theorem may be proved by the methods used for Theorem 13. It follows by Theorem 3 that the space is locally compact at P.

Note. The statements of Theorems 12 and 13 differ only in that in the hypothesis of the one the distributive property takes the place of local compactness in that of the other; they are stated for both spaces S and Hausdorff spaces. Theorems 12A and 13A are analogous generalizations of Theorems 12 and 13 respectively; but 12A is stated for a Hausdorff space, while 13A is stated for a space S. The question arises as to whether each of the Theorems 12A and 13A hold for both types of spaces. The author has not found the answer for the case of 13A; for 12A the answer is in the negative, as may be seen by considering a space T, when points and limit points are those of $P+K+\sum_{i=1}^{i=\infty} N_i$ of the proof of Theorem 13.

THEOREM 14. *If a Hausdorff space or a space S has the distributive property and each point in it has a monotonic family of neighborhoods, then the following properties hold for the space*: (1) *It satisfies the first countability axiom*; (2) *it is both a Hausdorff space and a space S*; (3) *every point set in it is separable*; *and* (4) *it is regular and locally compact.*

Proof. By Theorems 6 and 1 our space satisfies the first countability axiom; by Theorem 11 it is both a space S and a Hausdorff space; by Theorem 10 and Lemma I every point set in it is separable; by Theorem 13 it is regular; and by Theorem 3 it is locally compact.

Note. Theorems 14 and 15 may be regarded as a summary of results of this paper with regard to conditions necessary for the distributive property. Theorem 16 deals with sufficient conditions; Theorems 15 and 17 with necessary and sufficient conditions.

THEOREM 15. *If a space S (Hausdorff space) satisfies the first countability axiom and has the distributive property, then in order that one of its sub-spaces have the distributive property it is necessary and sufficient that the sub-space be regular and locally compact.*

Proof. The necessity follows from Theorem 14; the sufficiency from Theorem 5.

THEOREM 16. *A sufficient condition that a Hausdorff space have the distributive property is that it be locally compact and have the Lindelöf property, and that every closed point set in it be separable.*

Proof. By Theorem 2 our space satisfies the first countability axiom; by Theorem 12 it is regular; by Theorem 11 and Lemma I every point set in it is separable. The proof may be completed by following the methods for Theorem 9 of the author's first paper, p. 678. Theorem 17 follows from Theorems 14 and 16.

THEOREM 17. *In order that a Hausdorff space which has the Lindelöf property and in which every point has a monotonic family of neighborhoods should have the distributive property, it is necessary and sufficient that the space be locally compact and that every closed set in it be separable.*

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CONCERNING THE DECOMPOSITION AND AMALGAMATION OF POINTS, UPPER SEMI-CONTINUOUS COLLECTIONS, AND TOPOLOGICAL EXTENSIONS

BY

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Reprinted from the
Transactions of the American Mathematical Society
Vol. 49, No. 3, pp. 410-466
May, 1941

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1. Introduction. A necessary and sufficient condition that a space H Fréchet have the Borel-Lebesgue covering property is that each monotonic collection of closed point sets in it have a nonvacuous product(1). Thus, if such a space is not perfectly compact, there exists in it a monotonic collection of closed sets, E, with a vacuous product; E is an example of what we call a boundary element. With the help of E we may define a boundary point, $P(E)$; if $P(E)$ is added to the basic space and a suitable topology is introduced, the closures of the set of elements of E contain $P(E)$; and the deficiency of the elements of E, that their product is vacuous, no longer holds for the aggregate of their closures in the extended space. By adding to S an aggregate of boundary points which satisfy suitable conditions we achieve the embedding of S in a perfectly compact Hausdorff space (cf. Theorem 16.1).

In Chapter I we introduce the class of *point elements*; these include the boundary elements. We define the relation of the *intersection* of two point elements; in terms of this we give an ordering which makes the aggregate of all point elements a quasi-partially ordered system. A repetition of this algorithm gives the quasi-partially ordered system M of all collections of point elements. By a process identifying the equivalent elements of M we obtain a partially ordered system whose elements we call *portions* of our basic space S, or *S-portions*; these are our ideal points. A relation of an open set D in S and an ideal point P is expressed by saying that *D is an S-neighborhood of P*; the S-neighborhoods are used in topologizing collections of ideal points. The relation *P is an end of M* between the ideal point P and the closed point set of S, M,

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Presented to the Society, in three parts: September 9, 1937, under the title *Perfectly compact Hausdorff spaces in which a normal space may be embedded*; December 28, 1937, under the title *Concerning upper semi-continuous collections and the decomposition of points of normal spaces*; and September 6, 1938, under the title *The extension of homeomorphisms of normal spaces to topologically related spaces*; received by the editors July 18, 1938, and, in revised form, September 9, 1940.

(1) Cf. Moore, (II) (the bibliography is at the end of the paper). Kuratowski and Sierpiński, loc. cit., show that the Borel-Lebesgue property is equivalent to the condition that each infinite point set have a complete limit point. Fréchet, (II), calls this property *perfect compactness*; several years later Alexandroff and Urysohn, (I), call it *bicompactness*.

A collection of sets is said to be *monotonic* provided that if K and L are two of its elements, then either $K \supset L$ or $L \supset K$; Moore, loc. cit.

suggests an analogy between Carathéodory's theory of ends and ours in that both are concerned with methods of approach or accessibility. Our end-theory is used as a technique in developing that of the S-neighborhoods.

We shall now indicate some of the ideas underlying the methods which we discussed in the preceding paragraph. The theory of point elements and that of ends suggest our regarding a point not as a static entity but rather as that of a relation of the point to the remainder of space, that involves methods of approach. Secondly, a point need not necessarily be indivisible. We have a conception according to which our space S consists of a quantity of basic matter; this may be decomposed and put together in various ways, and the same applies for each portion of this matter. In particular, the points of S are subject to such operations. There exists a maximal portion, that of all the matter; there prove to be atomic portions.

These ideas may be described by means of our ordered system of ideal points; let "$A < B$" be the ordering relation of this system. Let P be an ideal point and M be a collection of ideal points; if both $P < M$ and $M < P$, we shall think of P as a *summation* or an *amalgamation* of M, and of M as a breaking up or a *decomposition* of P; we think of M and P as describing different structures of the *same* portion of the basic matter of S. Let X and Y be collections of S-portions and "$y = \alpha(x)$" be a transformation from X to Y such that for $y_1 \in Y$, y_1 is the amalgamation of the elements of $\alpha^{-1}(y_1)$(²); in proceeding from $\alpha^{-1}(y_1)$ to y_1 we are involved in a change of structure, only; and, thus, this transformation may be said to *preserve S-portions* or to *leave S-portions invariant.*

If we were concerned with order relations, only, the system of all our ideal points would afford a satisfactory basis for the amalgamation and decomposition theory. From a topological standpoint, however, we require systems of elements for which the S-portion preserving transformations, $y = \alpha(x)$, which describe the amalgamation and the decomposition processes, satisfy continuity conditions. As illustrations of systems which meet this requirement we refer to a result by Stone(³). Of particular interest is the case where X and Y are perfectly compact Hausdorff spaces; this case leads us to apply a mapping theory by Kolmogoroff, which, for our purposes, finds a summary in Theorem VIII, p. 98, Alexandroff and Hopf, loc. cit. The adaption of this theory to the S-portion preserving transformations leads to the introduction of a special class of ideal points, the *amalgamation points.* It is interesting to note that since in applications we are to use both order and continuity relations, such relations are used in defining these points. Chapter II is devoted mainly to a development of the properties of amalgamation points; such properties have striking analogies to those of closed point sets in perfectly compact Hausdorff spaces. For instance, a decomposition of an amalgamation point into atomic

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(²) The symbol, $\alpha^{-1}(y_1)$ means all elements of X, x_1, such that $y_1 = \alpha(x_1)$.

(³) Loc. cit., p. 476, Theorem 88; we discuss this in §§16 and 20.

S-portions is a perfectly compact set (cf. Theorem 14.4). In §15 this analogy is shown to be fundamental; here we show that the decomposition M of an amalgamation point P into regular amalgamation points is perfectly compact if and only if it is upper semi-continuous relative to S; we develop an elaboration of the Kolmogoroff-Alexandroff-Hopf theory of the S-portion preserving continuous transformations of such decompositions.

The sections following §15 are devoted to applying these results. In Chapter III we investigate systems of such applications. In §19 we consider the ordered system $\delta(P)$ of all perfectly compact decompositions of P into regular amalgamation points; the ordering "$X < Y$," which means that X may be mapped on Y by a continuous, S-portion preserving mapping $y = \alpha(x)$, makes $\delta(P)$ a complete lattice. If δ is a sublattice of $\delta(P)$, the sum of the elements of δ, L_δ(4), becomes a lattice of amalgamation points. The zero of δ, ω_δ, is the system of atomic elements of L_δ, and may be mapped on each element of δ. Such ω_δ's may be described as universal inverse mapping spaces relative to the elements of δ. Particularly important is the case for which P is the maximal amalgamation point; that is, P is the amalgamation of all our ideal points. Then we let $\delta(S) = \delta(P)$; in §20 we show that the completely regular spaces S are characterized by the property that there exist elements of $\delta(S)$ in which the decomposition of S has proceeded to elements at least as small as the points of S. For such a space we have the interesting subsystem of $\delta(S)$, $\delta = H(S)$, which consists of the images of all immediate extensions of S which are perfectly compact Hausdorff spaces. The zero of $H(S)$, $\lambda(S)$, is the topological image of the space considered by Stone in his Theorem 88 (loc. cit., p. 476). In §17 the space $\lambda(S)$ enters into an interesting characterization of the normality of S; this is given by the condition that each upper semi-continuous decomposition of S into a collection of closed point sets can be extended to a similar decomposition of $\lambda(S)$. Further, only the topological images of $\lambda(S)$ have this property.

In Chapter II we introduce the concept of a semi-completely normal space. For such a space the theory of Chapter III is especially simple; such a space is characterized by the property that the zero of $\delta(P)$, $\omega(P)$, is a set of atomic ideal points. Similarly, the normal space is characterized by the condition that the only boundary points which are elements of an $\omega(P)$ are atomic points. In Chapter II we give characterizations of various types of normal and regular spaces.

CHAPTER I. AN ORDERED SYSTEM OF IDEAL POINTS

At the beginning of the preceding section we discussed the methods and the underlying ideas which occur in the development of our theory of point elements and ideal points. It may be mentioned that methods similar to those used in this chapter may be used to extend a partially ordered system K to a complete partially ordered system, granting that K satisfies suitable conditions.

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(4) It may be necessary to add a zero to L_δ to make it a lattice; such is *not* the case for $\delta(P)$, or for δ.

2. **Concerning a topological background.** We refer the reader to treatises by Fréchet, Hausdorff, Menger, Moore, Sierpiński, and Alexandroff and Hopf (cf. Bibliography). Throughout the paper the symbol S will denote a basic Hausdorff space. We shall consider only spaces H, Fréchet (cf. Fréchet, (I), p. 186). Alexandroff and Hopf call such spaces T_1 spaces (loc. cit., p. 59).

DEFINITIONS. D 2.1. The symbol M_T', the *derived set of M relative to T*, means the set of all points of T which are limit points of $M \cdot T$. The symbol $\overline{M}_T$ denotes the *closure of M relative to T*; that is, $M \cdot T + (M \cdot T)_T'$.

D 2.2. If T and R are spaces (1) such that $T \supset R$ and (2) such that if M is a point set in R then $\overline{M}_R = R \cdot \overline{M}_T$, then we say that R is a *subspace* of T, that R is *embedded* in T, and that T is an *extension* of R; if, in addition, T is a Hausdorff space, it is a *Hausdorff extension* of R. If $T = \overline{R}_T$, T is an *immediate extension* of R (cf. Stone, loc. cit., p. 420).

D 2.3. If T is an immediate Hausdorff extension of S, the points of $T - S$ are called *frontier* points of S. If P is a point of T, $T \supset S$, D is an open set of S, and P does not belong to the closure in T of $S - D$, we call D an *S-neighborhood* of P. Two frontier points of S are said to *intersect* provided that each S-neighborhood of the one contains points in common with each S-neighborhood of the other.

3. **Partially ordered and quasi-partially ordered systems.** We refer the reader to a treatise by Birkhoff and an article by MacNeille for discussions, examples, and references to the literature (cf. Bibliography).

D 3.1. By a partially ordered system[5] is meant a system X in which an
<y" (*ordering*) relation "x ~~$\leqq$~~ y" is defined which satisfies the following conditions: (P_1) For $x \in X$, $x < x$; (P_2) if $x < y$ and $y < x$, then $x = y$; (P_3) if $x < y$ and $y < z$, then $x < z$. If the condition P_2 is omitted, then X is a *quasi-partially ordered system*. The notation "$x < y$" will be read "y includes x," "x precedes y," or "x is a part of y." If $x < y$ but not $y < x$, we say that x is a *proper* part of y.

D 3.2. If $x < y$ and $y < x$, we say that x and y are *equivalent* (*relative to the ordering* "$<$"). Note that P_2 rules out distinct equivalent elements. In this paper we shall have many examples of quasi-partially ordered systems for which P_2 does not hold.

D 3.3. An element of X is said to be *atomic*, if no element of X is a proper part of it.

D 3.4. An element of X is a *lower bound* of a subcollection of X, N, if it is a part of each element of N. The set of all lower bounds of N is called the

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[5] Cf. Hausdorff, (I), p. 139 and Birkhoff, pp. 5, 7.

intersection $I(N)$ of the elements of N. A greatest element of $I(N)$ is called a *greatest lower bound* of the elements of N. The terms *upper bound* and *least upper bound* have analogous definitions. A *zero* and a *unit* of X are, respectively, a lower and an upper bound of all the elements of X.

D 3.5. If two elements of X have a lower bound in X, they are said to *intersect* (in X).

D 3.6. A subcollection of X, N, is said to be *monotonic* if for $x\in N$ and $y\in N$ either $x<y$ or $y<x$.

D 3.7. If X is partially ordered and each subcollection of X has a greatest lower bound in X, X is a *complete multiplicative system*.

D 3.8. A *lattice* is a partially ordered system X such that any two of its elements have both a greatest lower bound and a least upper bound in X; if each subset of X has such bounds, X is *complete*.

4. **The point elements.** In §1 we discussed the origin of our concept of a boundary element. This discussion applies also to a decomposition point element which satisfies (A) of D 4.1. If $M=a(E)$ is a non-isolated point of S, (4) of D 4.1 guarantees that E assist in the decomposition of M according to the theory we are to develop. For Case (A) the elements of E are subsets of $S-a(E)$; this is desirable for technical reasons in expressing relations of a point element to its *S-neighborhoods* (cf. D 4.5 and D 11.1). The fact that the topological relations of $M=a(E)$ to point sets in S depend on subsets of $S-M$ gives reasons that $S-M$ should contain S-neighborhoods of E.

D 4.1. A *boundary element of* S is a nonvacuous collection E such that (1) each of its elements is a nonvacuous, closed point set of S, (2) the product of any two elements of E contains an element of E, and (3) the product of all elements of E is vacuous. A collection of sets E is called a *point element of* S if either (A) or (B) is satisfied: (A) Either E is a boundary element of S, or there exists a non-isolated point of S, M, such that E is a boundary element of the subspace of S, $S-M$, and the following condition holds: (4) each open set in S that contains M contains an element of E(6). (B) There exists an isolated point of S, M, which is an element of E; each element of E is a closed point set of S which contains M. If E satisfies (B) it is said to be *degenerate*.

For Case (B) E satisfies conditions (1), (2), and (4), and the product of all elements of E is M. For Case (A) the product of all elements of E is vacuous, and (4) implies (3).

D 4.2. Let $a(E)=M$ be the product of the closures in S of the elements of E. We say that $a(E)$ is *attached* to E, and conversely.

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(6) The two cases under (A) may be combined as follows; this is the form of (A) we shall use in our applications: There exists an M, which is either a non-isolated point of S or is the null set, such that (1) each element of E, e, is a nonvacuous, closed point set in the space $S-M$, and its closure in S is $M+e$; (2) the product of two elements of E contains an element of E; (3) the product of all elements of E is a subset of M; (4) if M is a point of S, any open set of S which contains M contains an element of e.

D 4.3. If $a(E)$ is a point of S, E is called a *decomposition point element*. If E is a boundary element, $a(E)$ is vacuous.

D 4.4. Two point elements are said to *intersect*, provided that each element of the one has points in common with each element of the other.

D 4.5. An open set in S which contains an element of a point element E is called an S-*neighborhood* of E. A point or a frontier point of S is said to *intersect* E, provided that each S-neighborhood of the point has points in common with each S-neighborhood of E. (Cf. D 2.3.)

D 4.6. If E and F are point elements and each point element which intersects E intersects F, we say that "$E<F$"; we read this "*E is a portion of F (relative to S).*"

D 4.7. Let the aggregate of all point elements be ordered by the relation "E is a portion of F." It is easy to see that P_1 and P_3 of D 3.1 are satisfied but that P_2 is not. Thus, this system is quasi-partially ordered.

EXAMPLES. E 4.1. Let S be the Euclidean plane, M be the origin, and P_n be the point with coordinates $(1/n, 0)$. Let $e_{nk}=(P_n, P_{n+k}, P_{n+2k}, \cdots)$, and $E_{jk}=(e_{jk}, e_{j+k,k}, e_{j+2k,k}, \cdots)$. Then the E's are decomposition point elements of S and are attached to M; also, they are boundary elements of $S-M$. E_{12} and E_{22} do not intersect, but each is a portion of E_{11}. E_{33} intersects each of E_{12} and E_{22}; none of these point elements is a portion of any of the others. E_{66} is a greatest lower bound of E_{22} and E_{33}; and E_{36} is a greatest lower bound of E_{12} and E_{33}.

E 4.2. Let S be a space with infinitely many points, all of which are isolated. Then all decomposition point elements are degenerate. Any monotonic collection of point sets whose elements have a vacuous product is a boundary element.

E 4.3. Let H be a continuum in the plane whose points have coordinates (x, y) which satisfy one of the following conditions: (1) $0 \leqq x \leqq 1$, $y=0$; (2) $x=1/k$ and $0<y\leqq 1/k$, where $k=1, 2, 3, \cdots$. Let F be the aggregate of all subcontinua of H which contain the origin and at least one point with a positive ordinate. Let M be the origin and E be the aggregate of all sets $f-M$, where $f \in F$. Let G be the aggregate of all sets obtained by reflecting elements of E in the X axis. Then E and G are equivalent point elements. However, *no element of either G or E is a subset of an element of the other.*

5. **The intersection of point elements; atomic elements.** Because of Theorem 5.1 the definitions which are given for the intersection of point elements in D 3.5 and D 4.4 are logically equivalent. An analogous state of affairs shows that such a consistency should not be considered obvious: Let K be a partially ordered system; define for K a new order by an application of Definitions D 3.5, D 4.6, and D 4.7; the new order need not be consistent with the basic order of K.

THEOREM 5.1. *In order that two point elements E and F should intersect, it* *is necessary and sufficient that there exist a point element which is a portion of both; if this condition is satisfied and* G *is the aggregate of all sets which are the product of an element of* E *with an element of* F, *then* (a) G *is a point element and a greatest lower bound of* E *and* F, *and* (b) $a(E)=a(F)=a(G)$.

Proof. Let H be a common portion of E and F. Since H intersects itself, it intersects each of E and F. Since E intersects H and H is a portion of F, E intersects F.

Conversely, let E and F intersect. Suppose that $a(E)$ is a point of S and $a(E)\neq a(F)$. By D 4.1, (3), there exists $f\in F$ such that f does not contain $a(E)$. Since $\bar{f}=f+a(F)$, $a(E)\in S-\bar{f}$; by D 4.1, (4), $S-\bar{f}\supset e\in E$; since $e\cdot f$ is vacuous, we are involved in a contradiction. Thus, either $a(E)$ and $a(F)$ are the same point of S, or both are the null set. If E and F are degenerate, the conclusion in (a) is obvious; cf. D 4.1, (B). Suppose, therefore, that neither is degenerate. Let $g_1=e_1\cdot f_1$ and $g_2=e_2\cdot f_2$ be elements of G, and $K=a(E)$. Then $g_1\cdot g_2=(e_1\cdot f_1)\cdot(e_2\cdot f_2)=(e_1\cdot e_2)\cdot(f_1\cdot f_2)\supset e_3\cdot f_3=g_3\in G$; thus, Condition (2) of D 4.1 is satisfied; clearly the same holds for (3) and (4). Because of (2) and (4) $\bar{g}_1\supset g_1+K$; the converse is true, since $\bar{e}_1=e_1+K$ and $\bar{f}_1=f+K$. Thus, G satisfies all the conditions of D 4.1, (A), and it is a point element. Let H be a point element which intersects G, $h\in H$, $e\in E$, and $f\in F$. Then $h\cdot(e\cdot f)$ is nonvacuous. It follows that each of $e\cdot h$ and $f\cdot h$ is nonvacuous. Thus, H intersects each of E and F, and G is a common portion of E and F.

Let X be a point element which is a common portion of E and F, and Y be a point element which intersects X. Let e, f, x, and y be elements, respectively, of E, F, X, and Y. Let Z be the aggregate of all products of an element of X by an element of Y; similarly define T in terms of E and Z. By the preceding paragraph Z is a point element and $Z<X$; since $X<E$, Z intersects E, and T is a point element. Then $T<Z<X$ and T intersects X. Since $x\cdot y\cdot e\in T$ and $X<F$, $f\cdot(x\cdot y\cdot e)$ is nonvacuous. Then $y\cdot(e\cdot f)$ is nonvacuous; since $e\cdot f\in G$, Y intersects G. Thus, $X<G$, and G is a greatest lower bound of E and F.

THEOREM 5.2. *No decomposition point element intersects a boundary element.*

THEOREM 5.3. *A degenerate point element is atomic.*

THEOREM 5.4. *If* M *is a collection of point elements and each finite subcollection of* M *has a lower bound, then* M *has a greatest lower bound.*

This lower bound may not belong to M. The theorem has an analogy to conditions for perfect compactness; cf. Moore, (II), and Fréchet, (I), p. 231.

Proof. Let H be the aggregate of all point elements E such that E is a greatest lower bound of a finite subcollection of M; if $m\in M$, $m<m$; thus, $m\in H$ and $H\supset M$. Let K be the sum of the elements of H, and e and f be elements of K. There exist $E\in H$ and $F\in H$ such that $e\in E$ and $f\in F$. There exist two finite subcollections of M, H_E and H_F, such that E and F are greatest

lower bounds, respectively, of H_E and of H_F. By our condition $T=H_E+H_F$ has a lower bound; let β be *any* lower bound of T. Then β is a lower bound of H_E, of E, of H_F, of F, and of G, where G is a greatest lower bound of E and F (by Theorem 5.1 E and F have a greatest lower bound). Since β is *any* lower bound of T, $G<E$, and $G<F$, then G is a greatest lower bound of T; then $G\in H$ and $K\supset G$. By Theorem 5.1 $e\cdot f\in K$. Thus, Condition (2) of D 4.1 is satisfied by K. By Theorem 5.1 $a(E)=a(F)=a(G)$. It follows readily that the other conditions of D 4.1 are satisfied. Since each element of M is a subset of K, K is a portion of each of these elements. Thus, K is a lower bound of M. If the point element F is a lower bound of M, it follows from the definition of H that F is a portion of each element of H. If a point element intersects F, it intersects each element of H, and thus intersects K. Thus, $F<K$, and K is a greatest lower bound of M.

THEOREM 5.5. *If E is a point element, there exists an atomic point element which is a portion of E.*

Proof. There exists a monotonic collection of point elements, M, of which E is an element, which is not a proper subcollection of any monotonic collection of point elements; cf. Hausdorff, (I), p. 140. Let F be a greatest lower bound of M, and G be a point element such that $G<F$ (cf. Theorem 5.4). Then $G+M$ is monotonic, $G\in M$, and $F<G$; since $G<F$, by D 3.3 F is atomic.

5A. **Historical.** In §§16, 17, and 20 we discuss the space $\lambda(S)=S+M$, where S is a completely regular space, M is a collection of regular boundary points which are the atomic elements of a certain system, and $\lambda(S)$ is perfectly compact. This space has been studied by Stone, Wallman, and Čech(7). Čech, on page 833, has a result which is equivalent to the following: In order that S be normal, it is necessary and sufficient that if F_1 and F_2 are mutually exclusive closed point sets in S, and $\beta\in\lambda(S)-S$, then β is not a limit point of both. An analogous condition on one of our ideal points β is necessary and sufficient that β be atomic(8); this is true even if S is irregular. By Theorem 16.1 it is characteristic of a normal space that the elements of $\lambda(S)-S$ be atomic boundary points. Thus, Čech encounters the atomic points only in the case of the normal space, while we have them at our disposal for any Hausdorff space. This difference seems to be characteristic of the difference of our methods; here Čech seems to follow Tychonoff and Urysohn. Stone's methods involve an extensive use of algebra, but seem to have elements in common with those of Čech.

Wallman's methods most nearly resemble ours. He constructs the points of $\lambda(S)$ by means of collections of sets; for the case of the boundary points he uses what we should call atomic boundary elements. For the other type of points, those corresponding to the points of S, he uses definitions which are not suitable for our decomposition theory. Since he does not consider such a theory, his definitions are adequate for his purposes.

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(7) Cf. bibliography. Stone's paper appeared in May, 1937. The author's first report before the Society was given in September, 1937; it included most of the results of the first two chapters of the present paper for the case of the normal and semi-completely normal spaces. The papers of Cartan, Wallman, and Čech appeared later.

(8) Compare the comment on Theorem 11.7 and Theorem 12.3.

Cartan, on the other hand, has a basis for a decomposition theory, based on *order*, resembling that developed in our §5; however, he fails to develop the topological applications, and has nothing like the order-continuity theory of our Chapters 2 and 3. His *filter bases* and *filtres* correspond to our point elements and composition points; he proves the existence of *ultra-filtres* which correspond to our atomic points. He has the theorem: In order that two filter bases should generate the same filter, it is necessary and sufficient that each element of the one should contain an element of the other. By Example E 4.3 such a theorem is not true for our point elements. Thus, formally his definitions are not quite adequate for our treatment.

For a domain in the plane Carathéodory has a theory of chains, ends, and prime ends, which correspond to our boundary elements, boundary points, and atomic boundary points. His treatment of accessibility corresponds to our sections 6 and 10.

6. **A Theory of accessibility.** The *overlapping* of a point element and a closed set in S, which we define in D 6.1, resembles the *intersection* of two point elements (cf. D 4.4). Similarly, Theorem 6.1 is an analogue of Theorem 5.1. The discussion at the end of §5A suggests an interpretation of the results of Theorem 6.1 according to which a point element involves methods of approach or of accessibility. A boundary element may be regarded as a "way" for escaping from the basic space; the corresponding boundary point serves as a barrier for such an exit, and may be approached by such a way. In D 6.2 we formulate these ideas.

D 6.1. If E is a point element, M is a point set in S, and each element of E has points in common with M, then E and M are said to *overlap*.

D 6.2. Let E and F be point elements and M be a point set in S. If each point element which intersects E also intersects F or overlaps M, respectively, we say that *E is a way in F*, or that *E is a way in M*, respectively. Clearly, the former is equivalent to the relation "$E<F$"; the latter is analogous.

Examples. E 6.1. Let S be the plane, M be a line, Q be a point of M, and E be the aggregate of all sets $I-Q$, where I is an interval on M and Q is its midpoint. Then E is a maximal common way of M and E, and also of S and E. If R_1 and R_2 are the two rays on M from Q, then any way common to R_1 and E fails to intersect any way common to R_2 and E. We might say that there are many more ways of approach to Q in S than there are in M; and there are more in M than in R_1.

E 6.2. In the notation of Example E 4.1 the point element E_{36} is a maximal common way of the set e_{33} and the point element E_{12}.

THEOREM 6.1. *Let M be a closed point set in S and E be a point element.* (1) *In order that there exist a way common to M and to E, it is necessary and sufficient that M and E overlap*; (2) *if this condition is satisfied and $E(M)$ is the aggregate $[e \cdot M]$, where $e \in E$, then $E(M)$ is a maximal common way of E and M and $M \supset a(E(M))$.*

The content of the theorem and its proof are similar to those of Theorem 5.1. In particular, replace the "F" and the "f" of that proof by "M," and replace "G" by "$E(M)$."

THEOREM 6.2. *If F is a portion of E and G is a way common to M and F, then G is a way common to M and E.*

7. **The quasi-partial ordering of collections of point elements.** As in §4 we first define *intersection* and then define *order* in terms of intersection. The definitions of the two sections 4 and 7 are similar and the system of §4 is a subsystem of the one we consider here. In §1 we have explained how the results of §§7 and 8 afford a basis for a summation or an amalgamation theory, and how those of §4 yield a decomposition theory.

D 7.1. If a point element intersects one of a collection of point elements, the point element and the collection are said to *intersect*. Two collections of point elements are said to *intersect* if one of them intersects an element of the other.

D 7.2. Let A be the aggregate whose elements are (1) the point elements and (2) the collections of point elements. If M and N are elements of A and each element of A that intersects M also intersects N, we say that M *is a portion of N* (*relative to S*), or $M < N$. Clearly, this relation is reflexive and transitive.

D 7.3. Let the relation "$M < N$" order the elements of A. Clearly D 7.3 and D 4.6 are equivalent if both M and N are point elements. Thus the ordered system of the point elements of D 4.7 is a subsystem of A.

EXAMPLE. E 7.1. Let $M = E_{11}$, $N = (E_{12}, E_{22})$, $K = (E_{14}, E_{24}, E_{34}, E_{44})$, and $L = (E_{14}, E_{34})$. Then M, N, and K are equivalent, and so are L and E_{12}. Also, L is a portion of M, but the converse is not true. L does not intersect E_{22}. L intersects (E_{14}, E_{24}) but neither of these is a portion of the other.

D 7.4. If M is a point element or a collection of point elements, let $\Sigma(M)$ denote the set of all point elements that are portions of M; let $\alpha(M)$ denote the set of atomic elements of $\Sigma(M)$.

THEOREM 7.1. *If M and N are collections of point elements, then M is a portion of N if and only if each point element that intersects an element of M also intersects an element of N.*

THEOREM 7.2. (1) *If N is a collection of point elements and either $M \in N$*

or $N \supset M$, *then* $M < N$; (2) *if* M *is a collection of point elements and each element of* M *is a portion of* N, *then* $M < N$; (3) $\Sigma(N) \supset N$.

Proof. Consider (2). Let $E \in M$, and F be a point element which intersects E. By Theorem 5.1 there exists a point element G which is a lower bound of E and F. Since $G < E < N$, $G < N$, by D 7.2 there exists $n \in N$ such that G and n intersect; since $G < F$, n intersects F. By Theorem 7.1 $M < N$. Clearly each element of N is a portion of N; the conclusion of (1) of the theorem follows from (2).

The following theorem shows that the ordering relation $<$ is equivalent to the aggregate-inclusion relation $\subset$, if applied to the sets $\Sigma(X)$ and $\alpha(X)$; cf. D 7.4. It brings out the importance of the atomic elements in the ordering theory.

THEOREM 7.3. *Let* M *and* N *each be point elements or collections of point elements*: (1) $M < \Sigma(M) < \alpha(M) < M$; (2) *the conditions* (a) $M < N$, (b) $\Sigma(N) \supset \Sigma(M)$, *and* (c) $\alpha(N) \supset \alpha(M)$ *are logically equivalent*; (3) *the conditions* (a) M *intersects* N, (b) *the product* $\Sigma(M) \cdot \Sigma(N)$ *is nonvacuous, and* (c) *the product* $\alpha(M) \cdot \alpha(N)$ *is nonvacuous are logically equivalent*; (4) $\Sigma(M+N) \supset \Sigma(M) + \Sigma(N)$, *and* $\alpha(M+N) = \alpha(M) + \alpha(N)$; *each of these four sets is equivalent in our ordering to* $M+N$.

Condition (3) is an analogue of Theorem 5.1. Cf., also, Theorem 7.4. Condition (3) brings out the equivalence of D 7.1 and D 3.5.

Proof. Let the point element E belong to M, and let F be a point element which intersects E. By Theorems 5.1 and 5.5 there exists an atomic point element G which is a lower bound of E and F. By Theorem 7.2 (1), $E < M$; since $G < E$, $G < M$, and $G \in \alpha(M)$. By Theorem 7.1 $M < \alpha(M)$. Since $\Sigma(M) \supset \alpha(M)$, $\alpha(M) < \Sigma(M)$; cf. Theorem 7.2 (1). By Theorem 7.2 (2), $\Sigma(M) < M$.

Let M be a portion of N and $X \in \Sigma(M)$; since $X < M < N$, $X < N$, and $X \in \Sigma(N)$; thus, $\Sigma(N) \supset \Sigma(M)$. Conversely, if $\Sigma(N) \supset \Sigma(M)$, it follows by part (1) and Theorem 7.2 (1) that $M < \Sigma(M) < \Sigma(N) < N$. Thus (2a) and (2b) are equivalent. In the same way we can show that (2a) and (2c) are equivalent.

If M and N intersect, there exist intersecting point elements E and F which belong to M and N, respectively (cf. D 7.1). By Theorems 5.1 and 5.5 there exists an atomic point element G which is a lower bound of E and F. By Theorem 7.2 (1), $E < M$ and $F < N$. It follows that $G < M$ and $G < N$, or, that $G \in \alpha(M) \cdot \alpha(N)$. Since $\Sigma(M) \cdot \Sigma(N) \supset \alpha(M) \cdot \alpha(N)$, (3a) implies each of (3b) and (3c). By D 7.1 and Theorem 5.1 each of these implies (3a).

By Theorem 7.2 (1), $\Sigma(M) \supset M$; the proof of part (4) follows with the help of this theorem and parts (1) and (2).

THEOREM 7.4. *Let* M *be a collection of point elements, and of aggregates of point elements*: (1) *The set of all the point elements which belong to elements of* M *is a least upper bound of* M; (2) *if the elements of* M *have lower a bound, the* *product of the sets* $\alpha(m)$ *and that of the sets* $\Sigma(m)$, *where* m *ranges over* M, *are greatest lower bounds of* M.

Proof. Let K be the product of all the sets $\alpha(m)$ and L that of all the sets $\Sigma(m)$. By our hypothesis and Theorem 5.5 neither K nor L is vacuous. By Theorem 7.2 (2), and Theorem 7.3 (1), each of K and L is a lower bound of the elements of M. Let X be a lower bound of M, and Y be an element of M. By Theorem 7.3 $\alpha(Y)\supset\alpha(X)$; it follows that $K\supset\alpha(X)$. Since $L\supset K$, it follows from Theorems 7.2 and 7.3 that $X<\alpha(X)<K<L$. The conclusion of (2) follows. Part (1) may be proved by similar methods.

8. **The S-portions, our ideal points.** If E is a nondegenerate point element and we remove a finite number of the elements of E, the remainder is a point element which is equivalent to E. Thus, the condition P_2 for partially ordered systems does not hold for the quasi-partially ordered system of collections of point elements A, which we considered in the preceding section. In our applications to topology we find it desirable to remove the ambiguity which is involved in having equivalent, non-identical elements. We achieve this by a procedure which is essentially that of identifying equivalent elements; this yields a partially ordered system(⁹); its elements are our ideal points. For reasons given in §1 we call these *S-portions*.

D 8.1. Let A be the ordered system of all point elements and collections of point elements (cf. D 7.2 and D 7.3). If $E\in A$, let $P(E)$ be the collection of all elements of A, X, such that $X<E$ and $E<X$. We call $P(E)$ an *S-portion* or an *ideal point*. Clearly, $E\in P(E)$; if $X\in P(E)$, then $P(X)=P(E)$; for, if $Y\in P(X)$, $E<X<Y$, and $Y<X<E$.

D 8.2. Let P and Q be S-portions, $E\in P$, and $F\in Q$. If $E<F$ according to D 7.2, we shall say that $P<Q$, or that *P is a portion of Q*; also, we shall say that $P<F$ and $E<Q$.

D 8.3. Let $K(S)$ be the system of all S-portions; let it be ordered by the relation $P<Q$ of D 8.2. It is obvious that $K(S)$ is a partially ordered system; in particular, Condition P_2 holds.

D 8.4. If one of the elements of an S-portion is a point element, the S-portion is called a *composition point*; if the point element is a boundary element or a decomposition point element, respectively, the point is called a *boundary point* or a *decomposition point*, respectively; cf. D 4.3. The theory of composition points corresponds to that of §§4 and 5, while that of S-portions in general requires that of §7. We shall see in §14 that the composition points are included among the amalgamation points; these are a special class of the ideal points, and they are a basis for our topological applications. We shall see, also, that regular points and frontier points of S may be identified with composition points; cf. Theorem 13.7.

EXAMPLES. E 8.1. Let E be a finite collection of point elements such that

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(⁹) Cf. Birkhoff, p. 7, Theorem 1.2.

if E_1 and E_2 are elements of E then $a(E_1)=a(E_2)$; cf. D 4.2. Then $P(E)$ is a composition point. This follows from Theorems 14.11 and 14.8; or, it may be proved directly.

E 8.2. Let P be a fixed point in the plane and R be a ray with initial point P. Let $I(n, R)$ be $I-P$, where I is the interval on R with endpoint P and length $1/n$. Let $E(R)=(I(1, R), I(2, R), \cdots)$, and let E be the set of all $E(R)$'s. Let $R_1, R_2, R_3, \cdots$ be an infinite sequence of distinct R's. Let $F_n=I(n, R_n)+I(n+1, R_{n+1})+ \cdots$; let $F=(F_1, F_2, F_3, \cdots)$. Then F is a point element but does not intersect any element of E; by Theorem 8.1 $P(F)$ does not intersect $P(E)$, and *not* $P(F)<P(E)$. Suppose that H is a point element and $H<E<H$; then each $E(R_n)$ intersects H, and F intersects H; cf. D 4.4. Since $H<E$, F intersects E, and we have a contradiction. Thus, H does not exist, and $P(E)$ is *not* a composition point.

E 8.3. If E contains a boundary element and a decomposition point element, $P(E)$ is not a composition point.

D 8.5. If M is a point or a point set let $\Sigma(M)$ and $\alpha(M)$ denote, respectively, all composition points and all atomic ideal points which are portions of M (cf. D 7.4). Let E_P denote an element of the ideal point P.

D 8.6. A *degenerate* composition point is one which has degenerate elements. (Cf. D 4.1.)

THEOREM 8.1. *Let P and Q be ideal points.* (1) $\Sigma(E_P)\in P$, $\alpha(E_P)\in P$, *and* $\Sigma(E_P)\supset E_P$. (2) *The following are equivalent*: (a) $P<Q$; (b) $\Sigma(E_Q)\supset\Sigma(E_P)$; (c) $\alpha(E_Q)\supset\alpha(E_P)$. (3) *The following are equivalent*: (a) *P intersects Q*; (b) *E_P intersects E_Q*; (c) *$\alpha(E_P)\cdot\alpha(E_Q)$ is nonvacuous.* (4) *If P intersects Q, $P(\alpha(E_P)\cdot\alpha(E_Q))$ is their greatest lower bound in $K(S)$.*

(Cf. D 3.5 and the theorems of §7.)

THEOREM 8.2. *An atomic S-portion is a composition point. A composition point is atomic if and only if its elements contain only atomic point elements.*

Proof. Let $F\in E_P$ and $G\in\alpha(F)$; by Theorem 7.2 $F<E_P$. Since $G<F<E_P$, $P(G)<P(E_P)=P$. If P is atomic, $P=P(G)$, $G\in P$, and $F<E_P<G$; since G is atomic, so is F. The sufficiency may be proved by similar methods.

THEOREM 8.3. *Two atomic S-portions do not intersect.*

THEOREM 8.4. *A degenerate S-portion is atomic.*

The latter follows from Theorem 5.3. Let P and Q be atomic points which intersect in X. Then $X<P$; since P is atomic, $P=X$; similarly, $Q=X$.

9. **Intersections and orderings of collections of points.** We extend the concepts of intersection and of order to the class of all points and all collections of points. Our definitions, methods, and results are similar to those of preceding sections. We introduce the concepts of the summation and the decomposi-

tion of points. Example E 9.2 gives an indication of the complicated relations that may hold in our theory; some of the results in it may seem paradoxical.

D 9.0. The term, a *real point* is to mean a *point of* S. Hereafter, if the term *point* is used without any explicit or implicit qualifications, it may mean a real, an ideal, or a frontier point; cf. D 2.3.

D 9.1. If a real point or a frontier point intersects(10) a point element which belongs to an element of an S-portion, the point and the S-portion are said to *intersect*. Two S-portions are said to *intersect* if they have a lower bound in the system of ideal points; cf. D 3.5. If a point intersects one of a collection of points, it and the collection are said to *intersect*; similarly define the *intersection* of two collections.

D 9.2. Let B be the aggregate whose elements are the points and the collections of points. If M and N are elements of B and each ideal point that intersects M also intersects N, we say that *M is a portion of N*, or that $M<N$. Clearly, this relation gives a quasi-partial ordering for B. The formulation of our definition is similar to the condition of Theorem 7.1. By Theorem 8.1 $K(S)$ is a subsystem of B (cf. D 8.2 and D 8.3).

D 9.3. If P is a point and M is a collection of points, $P<M$, and $M<P$, we say that P is a *summation* of the elements of M; if, in addition, no two elements of M intersect, M is said to be a *decomposition* of P. $P(M)$ means an ideal point which is a summation of the elements of M.

D 9.4. If M is the set of all point elements, $P(M)$ is the *maximal* S-portion. Clearly, the term is justified, and $P(M)$ is the summation of all composition points. It follows from Theorems 8.3 and 5.5 that the set of all atomic ideal points is a decomposition of $P(M)$.

Examples. E 9.1. Let S be completely regular and R be a perfectly compact Hausdorff space which is an immediate extension of S. Then R is a decomposition of the maximal S-portion; cf. Theorem 16.1. Let M and N be the sets of all decomposition points and of all boundary points, respectively. Then $S<M$ and $M<S$; however, M and N fail to intersect. (Cf. Theorem 5.2.)

E 9.2. Let S be the plane and M be the set of all points of S. If F is a closed set, let $E(F)$ be the set of all point elements X such that each element of X is a subset of F; let $Q(F)$ be the ideal point of which $E(F)$ is an element. Let H be the aggregate of all $Q(L)$'s, where L is a line; and let K be the set of all $Q(C)$'s, where C is the circumference of a circle with a positive radius. Since no point of S is isolated, all point elements in $E(C)$ and in $E(L)$ are nondegenerate; it follows from D 4.1, Theorem 5.1 and D 7.1 that $E(C)$ and $E(L)$ do not intersect; by Theorem 8.1 $Q(C)$ and $Q(L)$ do not intersect; by D 9.1 and D 9.2 H and K do not intersect. Hence, neither $H<K$ nor $K<H$. If P is a real point common to C and L it follows from D 4.5 that P intersects elements of $E(C)$ and $E(L)$; it may be shown that P intersects ~~X~~(C) and ~~X~~(L).

Q(C) Q(L)

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(10) Cf. D 4.5.

Q

Since ~~I~~(C) intersects L but does not intersect ~~I~~(L), L is not a portion of ~~I~~(L); similarly, C is not a portion of ~~I~~(C), and M is a portion of neither H nor K. By Theorem 10.3 $E(L)$ contains boundary elements, but $E(C)$ does not; it follows that $Q(C) < C$, but not $Q(L) < L$; similarly, $K < M$, but not $H < M$.

Because of the similarity of the ideas involved in §§7 and 9 we can prove theorems analogous to those of §7 for collections of points, and can use similar proofs. We give several theorems about summations and decompositions.

Theorem 9.1. *If P is an S-portion, $\alpha(P)$ is a decomposition of P, and P is a summation of $\Sigma(P)$; $P < \alpha(P) < \Sigma(P) < P$.*

(2) *If M is a collection of ideal points, there exists one ideal point, $P(M)$, which is the summation of the elements of M; and $P(M)$ is the least upper bound of M in the system of the ideal points.*

Proof. By D 8.5 $\Sigma(P) \supset \alpha(P)$ and each element of $\Sigma(P)$ is a portion of P. By the analogue of Theorem 7.2 for the case of points, $\alpha(P) < \Sigma(P) < P$. Let X and Y be ideal points such that $Y < X$ and $Y < P$. By Theorem 8.1 $\alpha(E_P) \cdot \alpha(E_X) \supset \alpha(E_Y)$. If $F \in \alpha(E_Y)$, F is an atomic point element; by Theorems 8.1 and 8.2 $P(F)$ is atomic, $P(F) < X$, and $P(F) < P$. Hence $P(F) \in \alpha(P)$, and X intersects an element of $\alpha(P)$. By D 9.2 $P < \alpha(P)$. By Theorem 8.3 $\alpha(P)$ is a decomposition of P. This completes the proof of (1).

Consider (2). Let E be the sum of all sets $\alpha(E_m)$, where $m \in M$. Then E is a collection of point elements; let $P = P(E)$. Since $E \supset \alpha(E_m)$, by Theorem 7.2 $\alpha(E_m) < E$; by Theorem 7.3 and D 8.2 $m < P$; by the analogue of Theorem 7.2 for the case of points, $M < P$. By Theorem 8.1 if the ideal point A intersects P then E_A intersects E. By Theorems 7.3 and 7.1 E_A intersects an element of E, λ; then λ belongs to one of the sets, $\alpha(E_m)$, and E_A intersects this set; by Theorem 8.1 A intersects m; by D 9.2 $P < M$. Let the ideal point Y be an upper bound of the elements of M. Then, for $m \in M$, $\alpha(E_Y) \supset \alpha(E_m)$ (cf. Theorem 8.1). Then $\alpha(E_Y) \supset E \supset \alpha(E_m)$. By Theorem 7.3 and D 8.2 $m < P < Y$. Thus, P is a summation of the elements of M and is a least upper bound of them. Since the system of ideal points is partially ordered, P is unique.

Theorem 9.2. *In the partially ordered system of ideal points each collection M which has a lower bound has one greatest lower bound, β_M; and $\alpha(\beta_M)$ is the product of all the sets $\alpha(m)$, where $m \in M$.*

Proof. Let Y be an ideal point which is a lower bound of the elements of M; K be the product of all the sets $\alpha(m)$; $\beta_M = P(K)$; and $m \in M$. If $\lambda \in \alpha(Y)$, then $\lambda < Y < m$, and $\lambda \in \alpha(m)$; thus $\alpha(m) \supset \alpha(Y)$, and $K \supset \alpha(Y)$. Let the ideal point X intersect Y; by Theorem 9.1 $Y < \alpha(Y)$, and by D 9.2 X intersects an element of $\alpha(Y)$; since this element belongs to K, it follows from D 9.2 that $Y < \beta_M$. Similarly, we can prove that $\beta_M < m$. By Theorem 9.1 $K = \alpha(\beta_M)$. We have established the conclusion.

10. Accessibility and ends. We develop a theory for the accessibility of S-portions; this involves an extension of the theory of §6, where we deal with point elements. The discussion and the examples which were given there are appropriate here. The ends may be regarded as barriers at the terminations of the corresponding ways. We borrow the term *end* from Carathéodory; in §5A we indicate similarities between his treatment and ours. Theorem 11.3 brings out a dualism of the theories of the ends and of the S-neighborhoods; both are applied in the topologization of our ideal points, and in the development of this topology. The concepts of the present section, of *overlapping* and of *being a common way*, are analogous, respectively, to those of *intersecting* and of being a *common portion*; this may be seen by comparing the results of Theorem 10.1 with those of Theorems 5.1, 7.1, 7.4, and 9.2.

D 10.1. If M is a set of real points, K is a collection of point elements, and M and some element of K overlap(11), we say that M and K *overlap*; also, $P(K)$ and M *overlap*; if $P(K)$ is a portion of the real or the frontier point Q, we say that M and Q *overlap*.

D 10.2. If P and Q are points and each S-portion that intersects P overlaps M (and intersects Q), we say that *P is an end of M (in Q)*. If P and Q are ideal points, $E \in P$, $F \in Q$, and P is an end of M in Q, we say that *E is a way to P (in F* and *in M)* (cf. D 6.2). Clearly, the latter implies that $P < Q$.

Examples and comment. Adopt the notation of E 4.1; the composition point $P(E_{36})$ is an end of the point set e_{33} in the point $P(E_{12})$ (cf. also E 6.2). Consider E 9.2; the point $P(F)$ may be characterized as the summation of all composition points which are ends of F, and it is an end of F; $E(F) \in P(F)$, and each point element of $E(F)$ consists of subsets of F; Theorem 10.1 shows that it is characteristic of an end of a closed set of S that it have an element E such that any point element which belongs to E consists of subsets of the closed set. Such is *not* the case for *all* the elements of an end; for, consider E 4.3; here $P(E)$ is an end of the x-axis, but no element of E is a subset of the x-axis. If, in E 9.2, we let C' and L' be the sums of all the $E(C)$'s and of all the $E(L)$'s, respectively, we have two ways in S which have no common way; the ends of these ways, $P(C')$ and $P(L')$, do not intersect.

If A is a point of a linear interval and M is a countable sequence of points on the interval which converges to A, according to our theory there exist many more methods of approach to A in the arc than there exist in the sequence. The end of the arc is to be regarded as vastly smaller than A, but is much greater than the end of the sequence; the latter is capable of further subdivision. Similar interpretations hold for the other examples.

THEOREM 10.1. *Let M be a closed point set in S and P be an S-portion.* (1) *In order that there exist an end of M in P, it is necessary and sufficient that M and P should overlap.* (2) *Let P and M overlap, $G \in P$, and $G(M)$ be the collection of all*

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(11) Cf. D 6.1.

$E(M)$'s(12)*, where E is a point element which belongs to G; let Q be the ideal point $P(G(M))$; then Q is the maximal end of M in P.*

Proof. Let the ideal point A be an end of M in P. Since A intersects itself, by D 10.2 A overlaps M; and $A<P$. Since A overlaps M, there exists $E_A \in A$ which overlaps M (D 10.1). Since $A<P$, by Theorem 8.1 $\Sigma(E_P) \supset E_A$; by D 10.1 M overlaps each of $\Sigma(E_P)$ and P. The necessity of the condition in (1) follows.

Conversely, suppose that P and M overlap; by D 10.1 there exist $G \in P$ and a point element X, which belongs to G, such that X overlaps M; by Theorem 6.1 $X(M)$ is a point element; thus, $G(M)$ and $Q=P(G(M))$ exist. Let Z be an ideal point which intersects Q. By Theorem 8.1 E_Z intersects $G(M)$; by D 7.1 there exist point elements E_1 and $E(M)$ which belong to E_Z and to $G(M)$, respectively, and which intersect. Since each element of $E(M)$ is a subset of M, it follows by D 4.4 that each element of E_1 has points in common with M. By D 6.1 and D 10.1 M overlaps each of E_1, E_Z, and Z. There exists an element of G, E, such that each element of $E(M)$ is a subset of an element of E; then E_1 intersects E; by D 7.1 E_Z intersects G; by Theorem 8.1 Z intersects P. Since, also, Z overlaps M, Q is an end of M in P (cf. D 10.2).

Let G_1 be any element of P, whatever. Since each element of $E(M)$ is a subset of an element of E, it follows from D 4.4 and D 4.6 that $E(M)<E$; by Theorem 7.2 $G(M)<G$, and thus $G(M)<G_1$; by Theorem 7.1 $E(M)$ intersects an element, K, of G_1; since each element of $E(M)$ is a subset of M, it follows that K overlaps M. By the results of the preceding paragraph it follows that $G_1(M)$ is nonvacuous; and, if $Q_1=P(G_1(M))$, then Q_1 is an end of M in P.

Let β be an end of M in P, and Y be an ideal point which intersects β. Then $\beta<P$; by Theorem 8.1 $\alpha(G)\supset\alpha(E_\beta)$, and $K=\alpha(E_\beta)\cdot\alpha(E_Y)$ is nonvacuous. Let $F\in K$; by Theorems 7.3 and 7.2 $F<E_\beta$; by D 8.2 $P(F)<\beta$; by D 10.2 $P(F)$ overlaps M; by the result of the preceding paragraph, F overlaps M; by Theorem 6.1 $F(M)$ exists; by arguments we have used before, $F(M)<F<E_\beta<G$. By D 7.2 $F(M)$ intersects an element E of G. Since each element of $F(M)$ is a subset of M, it follows by an argument we used in the next-to-the-last paragraph that $E(M)$ exists and $F(M)$ and $E(M)$ intersect. Since $\alpha(E_Y)\supset K$, $F\in K$, and $F(M)<F$, it follows by Theorems 7.2 and 7.3 that $F(M)<F<K<E_Y$, and that $F(M)\in\Sigma(E_Y)$. Thus, $\Sigma(E_Y)$ and $G(M)$ intersect, since their respective elements $F(M)$ and $E(M)$ intersect. By Theorem 8.1 Y and Q intersect. Thus, by D 9.2 $\beta<Q$. Since β is an end of M in P, Q is the maximal end of M in P. Since $Q_1<Q$, and $Q<Q_1$, $Q=Q_1$. Thus, Q is independent of the particular element G of P. We have established the truth of (2) and the sufficiency of the condition in (1).

THEOREM 10.2. *Let M be a closed point set in S, P be an end of M, and Q*

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(12) Cf. Theorem 6.1. For example, $E(M)$ is the set of all products $e\cdot M$, where $e \in E$.

be an ideal point which intersects P; there exists an end of M which is common to P and Q. If $Q<P$, Q is an end of M.

THEOREM 10.3. *In order that the closed point set in S, M, be perfectly compact in itself, it is necessary and sufficient that no boundary point be an end of M.*

Proof. Suppose that M is not perfectly compact in itself; there exists in M a monotonic collection of closed point sets, K, which have no point in common (cf. Moore, (II)). By D 4.1, K is a boundary element. By Theorem 10.1, $P(K)$ is an end of M.

Conversely, let M be perfectly compact in itself, and E be a boundary element. Let K be the aggregate of all products $e \cdot M$, where $e \in E$. If E overlapped M, no element of K would be vacuous; then the product of the elements of K would be nonvacuous (cf. Fréchet, (I), p. 231). Then the product of the elements of E would be nonvacuous; by D 4.1 this is impossible. Since E does not overlap M, there exists no end of M in $P(E)$ (cf. Theorem 10.1).

THEOREM 10.4. *If the point set M of S is perfectly compact in itself and P is a point and a limit point of M, there exists a composition point which is an end of M in P.*

Proof. If R is a neighborhood of P relative to M, let $e(R)=\overline{R}_M-P$. Let E be the aggregate of all the $e(R)$'s. Since M is a closed point set in S, $\overline{R}_M$ is closed in S, and $e(R)$ is closed in $S-P$. Let U be an open set in S which contains P; then $U \cdot M$ is a neighborhood of P relative to M. M may be regarded as a regular subspace of S (cf. Alexandroff and Hopf, p. 89, Theorem IX). It follows that $U \cdot M$ and U contain an element of E. If we let $P=a(E)$, E satisfies condition (4) of D 4.1. It may readily be verified that E satisfies the remaining conditions. Thus, E is a point element, and $Q=P(E)$ is a composition point. By Theorem 10.1 Q is an end of M.

Let the ideal point X intersect Q; by D 10.2 X overlaps M. By Theorem 8.1 E_X intersects E; by D 7.1 there exists a point element, F, which belongs to E_X and intersects E. Since each S-neighborhood of P contains an element of E, and each element of F has points in common with each element of E, it follows that F and P intersect; cf. D 4.5. By D 9.1 X intersects P; by D 10.2 Q is an end of M in P.

11. **The S-neighborhoods.** These are applied in the topologization of our ideal points. The condition of "being separated by S-neighborhoods" of D 11.2 is analogous to the separations which are involved in Hausdorff's Axiom D and in the definition of normality; it finds analogous applications; cf. Theorem 11.5. The end theory of the preceding section is closely related to the S-neighborhood theory; duality of the two is brought out in Theorem 11.3. An important characterization of the atomic points is given in Theorem 11.7.

D 11.1. An open set in S, D, is called an *S-neighborhood of M* if one of the following conditions holds: (1) M is a point or a subset of D; (2) M is a point element and D contains an element of M; (3) M is an ideal point, a collection of points, or an aggregate of point elements, and D is an S-neighborhood of each element of M. Thus, to see whether D is an S-neighborhood of the S-portion P, first apply (2) for each point element which belongs to some element of P, then apply (3) for each element of P; then apply (3) for P. (Cf. D 2.3.)

D 11.2. If U and V are mutually exclusive open sets in S which are S-neighborhoods of M and of N, respectively, we say that they *separate M and N (relative to S)*.

THEOREM 11.1. *If K is an S-portion or a collection of point elements, and D is an open set in S, then D is an S-neighborhood of K if and only if K and $S-D$ do not overlap.*

Proof. Let D be an S-neighborhood of the S-portion P, $E\in P$, and $\lambda\in E$. By D 11.1 D is an S-neighborhood of λ and of E; and there exists $\beta\in\lambda$ such that $D\supset\beta$. Then $\beta\cdot(S-D)$ is vacuous; by D 6.1, Theorem 10.1, and D 10.1, neither λ, E, nor P overlaps M. The converse may be established by reversing this argument.

THEOREM 11.2. *If E is an element of the ideal point P, then E and P have the same S-neighborhoods.*

Proof. Let D be an open set in S. By Theorem 10.1 if some element of P overlaps $S-D$, so does every other element. The conclusion follows from Theorem 11.1 and D 11.1.

THEOREM 11.3. *Let D be an open set in S and P be an ideal point.* (1) *In order that D be an S-neighborhood of P it is necessary and sufficient that $S-D$ have no end in P*; (2) *in order that P be an end of $S-D$ it is necessary and sufficient that D fail to be an S-neighborhood of any portion of P.*

Proof. If D is an S-neighborhood of P, by Theorem 11.1 P does not overlap $S-D$. By Theorem 10.1, $S-D$ has no end in P. A similar argument proves the converse.

Let P be an end of $S-D$ and X be a portion of P. Since X intersects P, by D 10.2 X overlaps $S-D$; by Theorem 11.1 D is not an S-neighborhood of X. Conversely, suppose that D is not an S-neighborhood of any portion of P. Let Z be an ideal point which intersects P. By Theorem 8.1 $P(\alpha(E_P)\cdot\alpha(E_Z))$ is a lower bound of P and Z. By our hypothesis and Theorem 11.1 $S-D$ overlaps $P(\alpha(E_P)\cdot\alpha(E_Z))$; by Theorem 10.1 and D 10.1 $S-D$ overlaps $\alpha(E_P)\cdot\alpha(E_Z)$, $\alpha(E_Z)$, and Z. By D 10.2 P is an end of $S-D$.

THEOREM 11.4. *Let D be an S-neighborhood of the ideal point X, and P be a non-isolated point of D; then $D-P$ is an S-neighborhood of X.*

Proof. If E is a point element which belongs to an element of X, by D 11.1 there exists $e_1\in E$ such that $D\supset e_1$. If $a(E)=P$, it follows from the fact that P

is non-isolated that $S-P\supset e_1$ (cf. D 4.1 and D 4.2). If $a(E)\neq P$ there exist elements of E, e_2 and e, such that $S-P\supset e_2$ and $e_1\cdot e_2\supset e$ (cf. D 4.1). Then $D-P\supset e$. By D 11.1 $D-P$ is an S-neighborhood of E and of X.

THEOREM 11.5. *In order that a real or a frontier point should fail to intersect a composition point, it is necessary and sufficient that the two points be separated by a pair of their respective S-neighborhoods.*

Proof. Let the real point P be separated from the composition point Q by U and V. Then by D 11.1 V is an S-neighborhood of any point element which belongs to an element of Q; by D 4.5 and D 9.1 P intersects neither the point element nor Q. Conversely, suppose that P and Q do not intersect, and let the point element E be an element of Q. By D 9.1 P and E do not intersect; by D 4.5 P and E can be separated by a pair of their respective S-neighborhoods, U and V; by Theorem 11.2 V is an S-neighborhood of the point Q.

THEOREM 11.6. *If each of M and N is an ideal point or a collection of ideal points and $M<N$, then each S-neighborhood of N is an S-neighborhood of M.*

Proof. Suppose that D is an S-neighborhood of N but not of M; there exists $P\in M$ such that D is not an S-neighborhood of P. By Theorem 11.3 $S-D$ has an end in P, say Q; then $Q<P$. Since Q intersects P and $M<N$, by D 9.2 Q intersects an element Z of N; let β be a lower bound of Q and Z. If $\lambda\in\beta$, $\Sigma(E_Z)\supset\lambda$ and $\Sigma(E_Z)\in Z$ (cf. Theorem 8.1). Since $Z\in N$, D is an S-neighborhood of Z, of $\Sigma(E_Z)$ and of λ (cf. D 11.1). Since λ is an arbitrary element of β, D is an S-neighborhood of β; since $\beta<Q$, by Theorem 11.3, (2), we are involved in a contradiction. Thus, D does not exist.

THEOREM 11.7. *An S-portion P is atomic if and only if it satisfies the following condition: If the sum of a finite number of open sets in S is an S-neighborhood of P, at least one of them is an S-neighborhood of P.*

If P is nondegenerate, the following is an equivalent condition: *If M and N are closed point sets in S and have at most a finite number of common points, then P is not an end of both M and N.*

Proof. Let D_1 and D_2 be open sets whose sum D is an S-neighborhood of P. Let $S-D_1=M_1$ and $S-D_2=M_2$. Suppose that neither D_1 nor D_2 is an S-neighborhood of P; by Theorem 11.3 M_1 and M_2 have ends in P. If P is atomic, these ends are P. Let E be a point element such that $E\in P$. There exists $e\in E$ such that $D\supset e$. By Theorems 6.1 and 10.1 $E(M_1)$ and $E(M_2)$ are point elements and are elements of P. Then $e\cdot M_1\in E(M_1)$ and $e\cdot M_2\in E(M_2)$. Since any two elements of P intersect, by D 4.4 $(e\cdot M_1)\cdot(e\cdot M_2)$ is nonvacuous. Since $D\supset e$ and $M_1\cdot M_2=S-D$, we have a contradiction. Thus, the condition is necessary.

Suppose that P is not atomic and D is an S-neighborhood of P. By Theorems 8.1, 8.2, and 8.3 there exist two atomic point elements, E and F, which belong to $\alpha(E_P)$ but do not intersect. By D 4.4 and D 11.1 there exist $e \in E$ and $f \in F$ such that $D \supset e+f$ and $e \cdot f$ is vacuous. Let $D_1 = D - \bar{e}$ and $D_2 = D - \bar{f}$. Since D_1 does not contain e, it is not an S-neighborhood of E or of P; similarly, D_2 is not an S-neighborhood of P. By D 4.1 and D 4.2 if $a(E) \neq a(F)$, then $\bar{e} \cdot \bar{f}$ is vacuous and $D_1 + D_2 = D$. By D 4.1 if $a(E) = a(F)$, then $e + a(F) = \bar{e}$, $f + a(F) = \bar{f}$, and $\bar{e} \cdot \bar{f} = a(F)$; since $e \cdot f$ is vacuous, $a(F)$ is non-isolated. Then $D_1 + D_2 = D - a(F)$, and by Theorem 11.4 this sum is an S-neighborhood of P. Thus, the condition is sufficient.

CHAPTER II. TOPOLOGICAL PROPERTIES OF THE IDEAL POINTS

In the first chapter we considered intersection and order; our treatment now requires the introduction of relations which involve continuity. In §12 we introduce a topology with the help of the S-neighborhoods. As a suggestion of problems we shall consider and of methods we shall use, we refer the reader to Alexandroff and Hopf, loc. cit., pp. 95–98; note particularly Theorem VIII. In this theorem they consider a perfectly compact Hausdorff space X, a decomposition $\sum A$ of X into mutually exclusive closed point sets, and a mapping $y = \alpha(x)$ of X into a space Y such that $\alpha(x_1) = \alpha(x_2)$ if and only if x_1 and x_2 are elements of the same element of the decomposition. They develop equivalent conditions, which include: (1) the transformation is continuous and Y is a perfectly compact Hausdorff space; (2) the transformation is continuous, and Y is a Hausdorff space; (3) the decomposition is upper semi-continuous.

In our Theorems 15.3, 15.4 and 15.5 we obtain extensive generalizations of the results of these authors. These results involve properties which restrict our attention to a special class of the S-portions, the amalgamation points. In §14 we develop a geometric theory of these points in which the amalgamation points have properties similar to those of the perfectly compact point sets in Alexandroff and Hopf's treatment. In Theorem 15.5 this analogy is strengthened by a kind of representation theory(13) in which the amalgamation points are identified with certain perfectly compact collections of amalgamation points, to which Alexandroff and Hopf's results may be applied. Corresponding to the set-theoretic treatment of upper semi-continuity of these authors our treatment requires the introduction of a geometric upper semi-continuity theory; cf. D 15.1. The latter part of the paper is devoted to the application of the basic theorems of §15.

In §13 we give characterizations of various types of regular and normal spaces in terms of the regular composition points. In §§16 and 17 we find that these characterizations prove useful in applying the results of §15. Such applications are particularly simple and extensive for the case of the semi-completely normal spaces, which we introduce in §13. For them the development of Chapter II has a completeness that is attained for more general spaces only in Chapter III.

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(13) Cf. Birkhoff, p. 76. Our representations involve not merely order but also continuity.

12. The topology of our points. We first give the following definition.

D 12.1. If P is a point and M is a point set, and each S-neighborhood of P is an S-neighborhood of infinitely many elements of M, then P is a *limit point* of M. Similarly, if P and M are a point element and a collection of point elements, respectively, P is called a *limit element* of M.

THEOREM 12.1. *The space $T(M)$, which is obtained by topologizing M by* D 12.1 *is a space H Fréchet. Let U be an open set in S and let the symbol $U(M)$ denote any nonvacuous subset of M consisting of all elements of M, with the possible exception of a finite number, of which U is an S-neighborhood. The collection of all such $U(M)$'s may be taken as a system of neighborhoods for defining limit points in $T(M)$.*

If P is the point and N is a subset of M, clearly P is a limit of N according to D 12.1 if and only if each $U(M)$ that contains P contains points of N distinct from P. By Fréchet, (1), pp. 185–186, $T(M)$ is a space H.

THEOREM 12.2. *If the composition point P neither intersects nor is a limit point of the set M of composition points, there exists an S-neighborhood of P which is not an S-neighborhood of any element of M.*

Proof. P has an S-neighborhood D, which is an S-neighborhood of at most a finite number of elements of M, $X_1, X_2, \cdots, X_m$. Let E and F_i be point elements which are elements of P and of X_i, respectively. There exist $e_i \in E$ and $f_i \in F_i$ such that $e_i \cdot f_i$ is vacuous. By D 4.1 there exists $e \in E$ such that $e_1 \cdot e_2 \cdot e_3 \cdots e_m \supset e$. Let $U=(S-\bar{f}_1)\cdot(S-\bar{f}_2)\cdots(S-\bar{f}_m)$. Then $U \supset e$ and U is an S-neighborhood of E and of P; but U is not an S-neighborhood of F_i or of X_i. Then $D \cdot U$ satisfies the conclusion.

THEOREM 12.3. *In order that a composition point be a limit point of a closed set of S, it is necessary and sufficient that the point be nondegenerate and contain an end of the set.*

Proof. Let M be the set, P be a point which satisfies the condition, and E be a point element which is an element of P. By Theorem 10.1 P and E overlap M; by D 6.1 each element of E contains points of M. By the hypothesis and D 8.6 E is nondegenerate; by D 4.1 no point of M is common to all elements of E; thus, each of these elements contains infinitely many points of M; by D 11.1 each S-neighborhood of P contains an element of E. Thus, P is a limit point of M.

If there exists no end of M in P, by Theorem 11.3 $S-M$ is an S-neighborhood of P. Then P is not a limit point of M. If P is degenerate by D 8.6 and D 4.1 $a(E)$ is an isolated point of S; then $a(E)$ is an S-neighborhood of E and of P, and P is not a limit point of M.

D 12.2. If M is a collection of point elements, let $a(M)$ be the collection of all real points which are attached to elements of M (cf. D 4.2). If M is a real point or a frontier point, let $a(M)$ be M or the null set, respectively. If E is a set of points, or is an ideal point, let $a(M)$ be the sum of all sets $a(m)$, where $m \in M$. The concept $a(M)$ is applied in the definition of regularity (cf. D 13.1). In Example E 9.2, $a(Q(F))=F$. The method of the definition involves an induction similar to that of D 11.1.

THEOREM 12.4. *Let P and Q be ideal points, $P<Q$, $E \in P$, and D be an S-neighborhood of P. Then $a(Q) \supset a(P)$, $a(P)=a(E)$, and $\overline{D}_S \supset a(P)$.*

Proof. By D 8.2 if $F \in Q$, $E<F$. If $X \in a(E)$, there exists $\lambda \in E$ such that $a(\lambda)=X$. By Theorem 7.1 there exists $\delta \in F$ such that λ intersects δ; by Theorem 5.1 $a(\lambda)=a(\delta)$. Thus, $a(F) \supset a(E)$. Since $a(Q)=\sum a(F)$, for $F \in Q$, and $a(P)=\sum a(E)$, for $E \in P$, it follows that $a(Q) \supset a(P)$. If $P=Q$, then $E<F$, and $F<E$; then $a(E) \supset a(F)$, and $a(F) \supset a(E)$; thus, $a(E)=a(F)$; since F is any element of $Q=P$, $a(E)=a(P)$. By D 11.1 D is an S-neighborhood of $\lambda \in E \in P$. If $X \in S-\overline{D}_S$, by D 4.1 and D 12.2 X is not an element of $a(\lambda)$, of $a(E)$, or of $a(P)$.

13. **Regular points and normal spaces.** Since we deal extensively with perfectly compact Hausdorff spaces of our ideal points, we find need of the concept "the regularity relative to S of an ideal point." In terms of this concept we characterize various types of regular and of normal spaces (Theorems 13.1 to 13.3). These characterizations are of interest, since the regularity of a Hausdorff space does not imply its normality. These characterizations yield methods for applications of the decomposition and mapping theory of §15; the most extensive results are given for the semi-completely normal spaces, which we introduce in this section. Normality proves to be a weaker condition than semi-complete normality; and the latter is weaker than complete normality. In Theorem 13.7 we show that regular real or frontier points may be replaced by regular composition points in questions which involve either continuity or order. This procedure gives a basis for applying our theory to questions concerning the embedding of S in other spaces, and to those involving mappings of S. In general our methods do not require the regularity or the normality of the basic space; they are applicable locally, so long as they deal with regular ideal points. For instance, we are in a position to study an infinitesimal regular portion of an irregular point. For the plane all composition points are regular.

D 13.1. A point P is *regular* (*relative to* S) provided that if D is an S-neighborhood of P there exists an S-neighborhood of P, R, such that $D+a(P) \supset \overline{R}_S$ (cf. D 12.2 and Alexandroff and Hopf, p. 68).

D 13.2. S is said to be *semi-completely normal* provided that if P is a point of S, then the subspace of S, $S-P$, is normal.

Clearly each completely normal space has this property; the plane is an example. In Theorem 13.4 we show that each semi-completely normal space is normal; an example by Tietze shows that the converse is not true (loc. cit., pp. 304–306). An example by F. B. Jones shows that *if the hypothesis of the continuum is valid*, then there exists a semi-completely normal space which is not completely normal. He has constructed a space which is normal (loc. cit., pp. 673–675). On page 674 he has a point set N, which is uncountable and contains no limit point of itself. By his Theorem 4 and the *continuum hypothesis* S is not completely normal. It is obvious from his construction of S that if P is a point of S then $S-P$ is normal.

THEOREM 13.1. (1) *In order that a point of* S *be regular, it is necessary and sufficient that it intersect no boundary point.* (2) *An irregular point of* S *intersects an irregular boundary point.*

Thus, a regular space is characterized by the property that it be a Hausdorff space and that none of its points intersect a boundary point.

Proof. Let P be irregular, and U be an open set containing P such that if R is an S-neighborhood of P, then $\overline{R}\cdot(S-U)$ is nonvacuous. If $M=\overline{U}-U$ were perfectly compact, it would be possible to separate P and M by a pair of open sets, U_1 and U_2(14); then $U\cdot U_1$ is an open set which contains P and is a subset of U; the closure of this set is a subset of U, since it contains no point of U_2 and $U_2\supset M$. Thus we are involved in a contradiction; it follows that there exists a collection of open sets, G, which covers M such that no finite subcollection of G covers M (cf. Kuratowski and Sierpiński, loc. cit.). For $X\in M$ let $U(X)$ and $W(X)$ be a definite pair of mutually exclusive open sets which contain X and P, respectively, such that $U(X)$ is a subset of some element of G. Clearly, no finite collection of the $U(X)$'s covers M. Let D denote the sum of a finite collection of the $U(X)$'s, $F=M-M\cdot D$, and E be the set of all F's. It follows that E is a boundary element; let it be an element of the boundary point β. Suppose that β is regular; by Theorem 11.4 $S-P$ is an S-neighborhood of β; since $a(\beta)$ is vacuous, by D 13.1 there exists an S-neighborhood of β, V, such that $S-P\supset\overline{V}$; then $S-\overline{V}$ is an open set containing P. Since V contains an element of E, $M-M\cdot V$ can be covered by a finite collection of the $U(X)$'s, say $U(X_1)$, $U(X_2)$, $\cdots$, $U(X_n)$. The sum of V and this finite collection is an open set which contains M. Let $\delta=U\cdot(S-\overline{V})\cdot W(X_1)\cdots W(X_n)$. Then δ is an open set which contains P, $\delta\cdot M$ is vacuous, and $U\supset\overline{\delta}$. Thus, the supposition that β is regular involves a contradiction. Suppose that P and β did not intersect; then by Theorem 11.5 β and P can be separated by a pair of their respective S-neighborhoods, V and W_1; by Theorem 11.2 and D 11.1 V contains an element of E; since $\overline{V}$ does not contain P we arrive at a contradiction as we did above. Thus, we have established the truth of (2) and the sufficiency of the condition in (1).

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(14) Follow the proof of Theorem IX, p. 89, Alexandroff and Hopf.

Let P be regular, E be a boundary element, and $\beta = P(E)$. By D 4.1 there exists $e \in E$ such that $P \in S - e$. Since e is closed in S, there exists an open set U, containing P such that $S - e \supset \overline{U}$. By D 11.1 and Theorem 11.2 $S - \overline{U}$ is an S-neighborhood of E and of β. By Theorem 11.5 P and β do not intersect. Thus, the condition is necessary.

THEOREM 13.2. *A Hausdorff space is normal if and only if each of its boundary points is regular.*

Proof. Let S be normal, E be a boundary element, and D be an S-neighborhood of the boundary point $P(E)$. By D 11.1 there exists $e \in E$ such that $D \supset e$. Since $S - D$ and e are closed and mutually exclusive, they can be separated by a pair of mutually exclusive open sets U and V. Since $V \supset e$, by Theorem 11.2 V is an S-neighborhood of $P(E)$; also $D \supset \overline{V}$.

Conversely, let the condition hold in S and M and N be mutually exclusive closed sets. By Theorem 13.1 each point of S is regular. For $P \in M$ let $D(P)$ be an open set which contains P such that $N \cdot \overline{D(P)}$ is vacuous. Let D denote any open set which is the sum of a finite number of the $D(P)$'s; and let E be the aggregate of all the sets $M - M \cdot D$. If one of the D's contains M, the complement of its closure contains N; if not, E is a boundary element. Since $S - N \supset M$, $S - N$ is an S-neighborhood of $P(E)$; by our condition there exists an S-neighborhood of $P(E)$, U, such that $S - N \supset \overline{U}$; then U contains an element of E, and there exists a D such that $U \supset M - M \cdot D$. Then $D + U \supset M$, and $N \cdot (\overline{D} + \overline{U})$ is vacuous. Thus, the complement of $\overline{D} + \overline{U}$ is an open set which contains N, and the condition is sufficient.

THEOREM 13.3. *A Hausdorff space is semi-completely normal if and only if each of its composition points is regular.*

Proof. Let S be semi-completely normal, β be a composition point, and E be a point element which is an element of β. If β is degenerate, then $a(\beta)$ is an isolated point of S and is an S-neighborhood of β; then β is regular (cf. D 8.6 and D 4.1). If $a(\beta)$ is a non-isolated point of S, then by D 13.2 the subspace $S - a(\beta)$ of S is normal; then β is a boundary point of this subspace; by Theorem 13.2 β is a regular boundary point of this subspace. It follows with the help of Theorem 11.4 that β is a regular decomposition point of S. If β is a boundary point of S let E be a point element, $E \in \beta$; let P be a point of S; and let F be the collection of all elements of E which do not contain P; it follows that F is a boundary element of S and of $S - P$; also, in S, $F < E$, $E < F$, and $F \in \beta$; by the argument we used above it follows that F is an element of a regular boundary point of $S - P$; and, hence, that β is regular relative to S.

Conversely, let all composition points of S be regular and let P be a point of S. By Theorem 13.2 S is normal and $S - P$ is regular. Methods analogous to those used for Theorem 13.2 show that $S - P$ is normal; by D 13.2 S is semi-completely normal.

THEOREM 13.4. *A semi-completely normal space is normal.*

By Theorem 13.3 all the boundary points are regular; the conclusion follows from Theorem 13.2.

THEOREM 13.5. *If F is a finite point set in a semi-completely normal space S, then $S-F$ is semi-completely normal.*

This may be proved with the help of Theorem 13.3 and the methods used in establishing that theorem.

THEOREM 13.6. *Let a regular point P be added to S so that the space $S+P$ is a Hausdorff space. If S is normal, so is $S+P$; in order that $S+P$ be semi-completely normal, it is necessary and sufficient that S have the same property.*

Thus a semi-completely normal space is characterized by the following property: If we either add or remove a finite number of regular points, or do both, and the result is a Hausdorff space, it is normal.

Proof. Let Q be a boundary point of the space $S+P$. Since P is a regular point, by Theorem 13.1 Q and P do not intersect relative to $S+P$; by Theorem 11.5 there exist in $S+P$ mutually exclusive open sets U and V which are $(S+P)$-neighborhoods of Q and P, respectively. Let E be a point element of $S+P$ such that no element of E contains P and that $E\in Q$; then E is a point element of S. Since S is normal and U is also an S-neighborhood of E, it follows from Theorem 13.2 that there exists an S-neighborhood of E, W, such that $U\supset\overline{W}$. Since the closures of W in S and in $S+P$ are identical, it follows that Q is a regular boundary point of $S+P$. By Theorem 13.2 $S+P$ is normal.

THEOREM 13.7. *Let P be a real, a frontier, or a composition point which is regular; if $a(P)$ is a non-isolated point of S, let $F(P)$ be the aggregate of all sets, $\overline{D}_S-a(P)$, where D is an S-neighborhood of P; otherwise, let $F(P)$ be the set of all the $\overline{D}_S$'s. Let $Q(P)$ be the composition point of which $F(P)$ is an element. Then* (1) *$a(P)=a(Q(P))$, $P<Q(P)$, and $Q(P)<P$;* (2) *if P is a limit point of a point set, then so is $Q(P)$; and conversely;* (3) *each S-neighborhood of P is an S-neighborhood of $Q(P)$; if D is an S-neighborhood of $Q(P)$, either D or $D+a(P)$ is an S-neighborhood of P;* (4) *$Q(P)$ is regular.*

Thus, $Q(P)$ is equivalent to P both from the point of view of order and of continuity. The topological applications of our theory require such a twofold equivalence; we discuss this in the introductions to Chapter II and to §§14 and 15.

Proof. Let P be a composition point and E be a point element such that $E\in P$. Since each element of $F(P)$ contains an element of E, $E<F(P)$ (cf. D 4.6). Let H be a point element which does not intersect E; there exist $e\in E$ and $h\in H$ such that $e\cdot h$ is vacuous. Then $S-\bar{h}\supset e$; by Theorem 11.2 $S-\bar{h}$

is an S-neighborhood of P. By D 13.1 there exists an S-neighborhood of P, U, such that $a(P)+(S-\bar{h})\supset \overline{U}_S$. Then $\overline{U}_S-a(P)$ or $\overline{U}_S$ is an element of $F(P)$, and contains no point of h. By D 4.4 $F(P)$ and H do not intersect. (It may readily be verified by D 4.1 that $F(P)$ is a point element.) It follows that $F(P)<E$ (cf. D 4.6). Since also $E<F(P)$, it follows by D 8.2 that $P<Q(P)$ and $Q(P)<P$; thus, $P=Q(P)$.

Next, let P be a real point or a frontier point. Clearly, $F(P)$ satisfies the conditions of D 4.1 and is a point element. Let X be a composition point which does not intersect P. By Theorem 11.5 P and X can be separated by a pair of their respective S-neighborhoods, U and V. There exists an S-neighborhood of P, W, such that $U\supset \overline{W}_S$. Then U contains an element of $F(P)$. By D 11.1 V contains elements of each point element which belongs to X; by D 4.4, D 7.1, and Theorem 8.1 $Q(P)$ and X do not intersect. By D 9.2 $Q(P)<P$. Conversely, let X be an ideal point which does not intersect $Q(P)$; by Theorem 8.1, if G is a point element belonging to X, G does not intersect $F(P)$. By D 4.4 there exist $f\in F(P)$ and $g\in G$ such that $f\cdot g$ is vacuous. There exists an S-neighborhood of P, D, such that either $f=\overline{D}_S$ or $f=\overline{D}_S-P$. By D 4.1 $S-\overline{D}_S$ contains g and is an S-neighborhood of G. By D 4.5 and D 9.1 P intersects neither G nor X. By D 9.2 $P<Q(P)$.

Since P is regular, each of its S-neighborhoods contains an element of $F(P)$ and is an S-neighborhood of $Q(P)$. If D is an S-neighborhood of $Q(P)$, $D\supset f\in F(P)$ (D 11.1 and Theorem 11.2). Then $f+a(P)=\overline{R}_S$, where R is an S-neighborhood of P and of $Q(P)$. Then $D+a(Q(P))\supset \overline{R}_S\supset R$, and $Q(P)$ is regular; since $a(Q(P))=a(P)$, either $D+a(P)$ or D is an open set containing R, and is an S-neighborhood of P.

If P is a non-isolated point of S, the conclusion of (2) follows from (3), (4), D 12.1, and Theorem 11.4. For the other cases, P and $Q(P)$ have the same S-neighborhoods, and (2) follows.

14. **The amalgamation points.** Suppose that the regular space T is an immediate extension of S. By Theorem 13.7 we can replace each point of S by an equivalent decomposition point, and each point of $T-S$ by a boundary point, and thus obtain a space T_1 of composition points which is topologically equivalent to T. We now introduce the amalgamation points in order to make more general applications, such as those we discuss in the introductions of Chapter II and of §13. These applications require that the amalgamation points include the composition points and that they have special properties which do not hold for all ideal points. These properties are similar to those of closed point sets in perfectly compact Hausdorff spaces. Thus, in Theorems 14.3, 14.4, and 14.5 we establish for amalgamation points a property analogous to the Borel-Lebesgue covering property; this analogy is emphasized in Theorem 15.3. Theorem 14.11 corresponds to the theorem that a nonvacuous product of perfectly compact sets is perfectly compact. In Theorems 14.6 and 14.7 we establish a condition for the separation of non-intersecting amalgama-

tion points; this suggests the condition for the separation of closed sets, which characterizes the normality of a Hausdorff space.

In Theorems 14.12 and 14.13 we add to the characterizations of the normal and the semi-completely normal spaces. For such spaces we obtain an extensive knowledge concerning the atomic elements of the developments of §§15 and 16. For the Hausdorff spaces in general such a completeness is attained in Chapter III only after the development of a theory of multiplicative systems and lattices of regular amalgamation points. Theorems 14.10 and 14.11 afford a basic technique for this lattice theory. The examples at the end of this section give indications of the extent to which our results may find applications in irregular spaces.

D 14.1. An S-portion, P which satisfies the following condition is called an *amalgamation point*: If $E \in P$ and X is a point element which is a limit element(15) of E, then X and E intersect. In the first chapter we defined *order* in terms of *intersection*. Since both order and continuity relations are used in our topological applications of the amalgamation points, it may be noted that the definition involves both order and continuity.

D 14.2. If M is a collection of points, P is an amalgamation point, $P < M$, and $M < P$, we say that P is the *amalgamation* of the elements of M. That is, the *summation* of D 9.3 is called an *amalgamation* if the result of the summation process is an amalgamation point.

Examples. E 14.1. Let S be the plane, E be the set of all decomposition point elements, and F be the set of all boundary elements. By Theorems 13.7, 13.1 and 5.2, E does not intersect F. Clearly, each element of F is a limit element of E. Thus, $P(E)$ is not an amalgamation point. If $X \in E$, there exists an S-neighborhood of X, R, whose closure is perfectly compact in itself; by Theorems 10.3 and 11.3 $S - \overline{R}$ is an S-neighborhood of each boundary element. Thus, X is not a limit element of F, and $P(F)$ is an amalgamation point.

E 14.2. The point $Q(F)$ of E 9.2 is an amalgamation point; it is the amalgamation of all ends of F.

The point of E 8.3 is an amalgamation point but is not a composition point. The point $P(E)$ of E 8.2 is not an amalgamation point; here F is a limit element of E but does not intersect E.

Theorem 14.1. *An S-portion P is an amalgamation point if and only if the following condition holds: If the composition point X is a limit point of the collection of composition points M and $P < M < P$, then X intersects M and P.*

It may be proved with the help of D 14.1 and Theorems 11.2, 8.1, and 9.1.

Theorem 14.2. *A composition point is an amalgamation point.*

Proof. Let E and F be non-intersecting point elements; by D 4.4 there exist $e \in E$ and $f \in F$ such that $e \cdot f$ is vacuous. Then $R = S - \bar{e} \supset f$. If the point element G belongs to an element of $P(E)$, it intersects itself and E (cf. D 8.1 and Theorem 7.1). If $g \in G$, $e \cdot g$ is nonvacuous, R does not contain g, and R is not an S-neighborhood of G. Thus, F is not a limit element of any element of $P(E)$. By D 14.1 $P(E)$ is an amalgamation point.

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(15) Cf. D 12.1, and D 7.1.

D 14.3. If M is a point set, G is a collection of open sets in S, and for $m \in M$ there exists $g \in G$ such that g is an S-neighborhood of m, then G is said to *cover* M (*relative to* S).

THEOREM 14.3. *Let M be an amalgamable collection of ideal points and let G cover M relative to S. Then there exists a finite subcollection of G such that the sum of its elements is an S-neighborhood of M.*

Note that the conclusion does *not* state that the finite subcollection covers M relative to S. The latter is the type of conclusion one has where the Borel-Lebesgue property holds. Theorem 14.3 is used in establishing the Borel-Lebesgue property for certain decompositions of an amalgamation point (cf. Theorems 14.4, 14.5, and 15.3).

Proof. Suppose that the conclusion does not hold for G. Let N be the aggregate of all atomic ideal points which are portions of elements of M. If R is an open set of S, let $N(R)$ denote the aggregate of all elements of N of which R is an S-neighborhood. Let H be the collection of all R's such that $N(R)$ is covered relative to S by a finite subcollection of G, say $G(R)$. Let the symbol L denote the complement in S of an element of H; let E be the aggregate of all L's, and let K be the product of all the L's.

It follows from Theorems 8.1 and 9.1 that if an ideal point intersects an element of either M or N, then it intersects an element of the other; by D 9.2, $M < N$ and $N < M$; by Theorem 11.6 each S-neighborhood of M is an S-neighborhood of N, and conversely. Since each element of N is a portion of an element of M, it follows by Theorem 11.6 that G covers N; if $R \in G$, then R covers $N(R)$ relative to S; thus $R \in H$, and $H \supset G$. Suppose that D_1 is the sum of the elements of a finite subcollection of H, say H_1, and D_1 is an S-neighborhood of M; then D_1 is an S-neighborhood of N; by Theorem 11.7 H_1 covers N relative to S. If $R_i \in H_1$, $G(R_i)$ covers $N(R_i)$ relative to S. Since N is the sum of the finite collection of $N(R_i)$'s, N is covered by the sum of the sets $G(R_i)$; this sum, G^*, is a finite subcollection of G. If D^* is the sum of the elements of G^*, D^* is an S-neighborhood of N and of M. This is contrary to our supposition concerning G. Thus, D_1 and H_1 do not exist. Thus, the hypothesis of our theorem holds for each of G and H, but the conclusion holds for neither.

Suppose that an element of E, L, were vacuous; then $S - L = S$ would be an element of H; then $N = N(S)$, and N is covered by a finite subcollection of G, $G(S)$; since $H \supset G$ we are involved in a contradiction (cf. the preceding paragraph). Thus, L does not exist. If R_1 and R_2 are elements of H, it follows from Theorem 11.7 that $N(R_1+R_2)=N(R_1)+N(R_2)$. Since $S-(R_1+R_2)=(S-R_1)\cdot(S-R_2)$, it follows that the product of two elements of E is an element of E.

Let $P\in K$. Suppose that $L_1\in E$ and P is not a limit point of L_1. There exists an open set in S, D, which contains P but contains no element of L_1-P. Then $S-L_1\supset D-P$ and $N(S-L_1)\supset N(D-P)$. Thus, $D-P\in H$ and $N(D-P)$ is covered relative to S by a finite subcollection of G. Since $P\in K\cdot D$, $S-D$ is not an element of E, D is not an element of H, $N(D)$ is not covered by a finite subcollection of G, and $N(D)\neq N(D-P)$. By Theorem 11.4 if P were non-isolated, we should have $N(D)=N(D-P)$; thus, P is isolated. By Theorem 8.4 there exists one atomic ideal point, β, such that $P=a(\beta)$, and β is degenerate (cf. D 12.2 and Theorem 13.7). Thus, $N(D)$ is either $N(D-P)$ or $\beta+N(D-P)$; in either case it can be covered by a finite subcollection of G, and $D\in H$. Therefore, we are involved in a contradiction, and L_1 does not exist.

Let $P\in K$. For R an S-neighborhood of P and $L\in E$ let $F(R, L)=(L-P)\cdot\overline{R}_S$, and let F be the aggregate of all $F(R, L)$'s. Since P is a limit point of each L, the product of two R's is an R, and the product of two L's is an L, it follows that the elements of F satisfy conditions (2) and (1) of D 4.1; if they also satisfy (4) they satisfy (3), and F is a decomposition point element. Suppose that (4) is not satisfied, and that U is a definite S-neighborhood of P which contains no element of F. Let T be the aggregate of all sets, $(\overline{U}-U)\cdot F(R, L)$, where $F(R, L)\in F$, and $U\supset R$. By the definition of U no element of T is vacuous. Since the product of two elements of F is an element of F, the product of two elements of T is an element of T. Let $Y\in\overline{U}-U$; since S is a Hausdorff space, there exist in it mutually exclusive open sets, V and W, which contain Y and P. If $R=U\cdot W$ and $L\in E$, then $(\overline{U}-U)\cdot F(R, L)$ does not contain Y; thus, the product of the elements of T is vacuous; by D 4.1 T is a boundary element. Similarly, if K is vacuous, E is a boundary element. Let Σ denote any of the three F, T, or E which is a point element; we have shown that at least one of them satisfies this condition. Σ has the property that if $L\in E$ then L contains an element of Σ. Let β be the composition point of which Σ is an element.

Let $X\in N$ and let Y be a point element which is an element of X (cf. Theorem 8.2 and D 8.4). There exists $g_X\in G$ such that g_X is an S-neighborhood of X (for, G covers N relative to S). Since $H\supset G$, $S-g_X\in E$, $S-g_X$ contains an element of Σ, and g_X contains an element of Y; by D 4.4 Σ does not intersect Y; by Theorem 8.1 β does not intersect X. Since $N<M<N$ and M is amalgamable so is N. By Theorems 9.1 and 14.1 β is not a limit point of N. By Theorem 12.2 there exists an S-neighborhood of β, D, which is an S-neighborhood of no element of N. There exists $\Sigma_1\in\Sigma$ such that $D\supset\Sigma_1$. Since $N(D)$ is vacuous, $D\in H$, $S-D\in E$, and $S-D\supset\Sigma_2$, where $\Sigma_2\in\Sigma$. Since $\Sigma_1\cdot\Sigma_2$ is vacuous, we are involved in a contradiction.

THEOREM 14.4. *An amalgamable collection of atomic ideal points has the Borel-Lebesgue covering property and is perfectly compact in itself.*

By Theorems 14.3 and 11.7 the set has the covering property. It is perfectly compact in itself; cf. Theorem 12.1 and Kuratowski and Sierpiński, loc. cit., Bibliography.

THEOREM 14.5. *The aggregate of all atomic ideal points is amalgamable and is perfectly compact in itself. The amalgamation of this collection is regular, and it is the maximal ideal point.*

Proof. Let M be the collection. By Theorem 9.1 the summation of the elements of M, β exists. If P and Q are intersecting ideal points, by Theorem 9.2 $\alpha(P)\cdot\alpha(Q)$ is nonvacuous; then $M\supset\alpha(P)\cdot\alpha(Q)$; by Theorems 9.2 and 9.1 Q intersects β and M; by D 9.2 $P<M$. By Theorem 14.1 β is an amalgamation point. Clearly, $a(\beta)$ consists of all points of S (cf. D 12.2 and D 9.4). By D 13.1 β is regular.

THEOREM 14.6. *Let P be a real point, Q be an amalgamation point, and M be a collection of composition points such that $Q<M<Q$. The following conditions are equivalent*: (1) *P and Q do not intersect*; (2) *no element of M intersects P*; (3) *P and Q can be separated by S-neighborhoods.*

Proof. Let $X\in M$ and E be a point element which belongs to some element of X. Since $X<Q$, by Theorem 8.1 E belongs to an element of Q. By D 9.1 if X intersects P, then so does Q; thus condition (1) implies condition (2). By condition (2) and Theorem 11.5 X and P can be separated by a pair of their respective S-neighborhoods, $U(X)$ and $V(X)$. By Theorem 14.3 there exists a finite collection of the $U(X)$'s such that the sum of these, U, is an S-neighborhood of M. If V is the product of the corresponding $V(X)$'s, V is an S-neighborhood of P, and $U\cdot V$ is vacuous. By Theorem 11.6 U is an S-neighborhood of Q. Thus, (2) implies (3). By D 11.1, D 11.2, D 9.1 and D 4.5, (3) implies (1).

THEOREM 14.7. *If P and Q are amalgamation points, one of which is regular, the following are equivalent*: (1) *P does not intersect Q*; (2) *P intersects no element of $\alpha(Q)$*; (3) *P and Q are separated by S-neighborhoods.*

Proof. By D 11.1, D 11.2, D 7.1, and D 4.4, and Theorem 8.1, (3) implies (1). By Theorem 9.1, (1) and (2) are equivalent; to establish (3) we shall consider the case of Q being regular. Let $X\in\alpha(P)$, $Y\in\alpha(Q)$, and E and F be point elements such that $E\in X$ and $F\in Y$ (cf. Theorem 8.2). By Theorem 8.1 E does not intersect F; there exist $e(Y)\in E$ and $f\in F$ such that $f\cdot e(Y)$ is vacuous. Then by D 4.1 $D(Y)=S-\overline{e(Y)}_s\supset f$, and $D(Y)$ is an S-neighborhood of F and of Y. By Theorems 9.1 and 14.3 there exist $Y_1, Y_2, \cdots, Y_n$ such that $D=D(Y_1)+D(Y_2)+\cdots+D(Y_n)$ is an S-neighborhood of $\alpha(Q)$ and of Q (cf. Theorem 11.6). By D 13.1 there exists an S-neighborhood of Q, R, such that

$D+a(Q)\supset\overline{R}_S$. By Theorem 9.1 $Q<\alpha(Q)<Q$; the methods of the proof of Theorem 12.4 show that $a(Q)=a(\alpha(Q))$. If each element of E contained points of $a(Q)$, by D 4.1, D 11.1, and D 12.2 X would be a limit point of $\alpha(Q)$; by Theorem 14.1 X intersects $\alpha(Q)$ and Q. By (1) this is impossible, and there exists $e_0\in E$ such that $a(Q)\cdot e_0$ is vacuous. By D 4.1 there exists $e\in E$ such that $e_0\cdot e(Y_1)\cdot e(Y_2)\cdot\cdot\cdot e(Y_n)\supset e$. Then $e\cdot(D+a(Q))$ is vacuous; $S-\overline{R}_S$ contains e and is an S-neighborhood of E and X; X and Q can be separated by S-neighborhoods. The conclusion for (3) follows by methods used for Theorem 14.6.

THEOREM 14.8. *A regular amalgamation point which is a portion of either a decomposition point or of a collection of boundary points is a composition point.*

Proof. Let P be the amalgamation point and Q be a decomposition point such that $P<Q$. Let $E\in P$; let F and G be point elements such that $F\in Q$ and G belongs to E; let $F(P)$ and $Q(P)$ be defined as in Theorem 13.7. By Theorem 7.2 and D 8.2 $G<E<F$. By D 7.1 G intersects F. By Theorems 5.1 and 12.4, and D 12.2, $a(Q)=a(F)=a(G)=a(E)=a(P)$. If R is an S-neighborhood of $a(P)$, it is an S-neighborhood of F, Q, and P (cf. D 4.1, D 11.1, and Theorem 11.6). Since P is regular, it has an S-neighborhood U such that $R+a(P)\supset\overline{U}_S$. Then $\overline{U}_S$ or $\overline{U}_S-a(P)$ is an element of $F(P)$. By D 4.1 $F(P)$ is a point element, and $a(P)=a(F(P))$.

Consider the second hypothesis for P. Let $X\in S$, $R=S-X$; by an argument similar to that of the preceding paragraph, it follows that $a(P)$ is vacuous, and $F(P)$ is a boundary element.

Since each element of $F(P)$ contains an S-neighborhood of P, each element of $F(P)$ contains elements of each point element which belongs to E. Thus, $E<F(P)$, and $P<Q(P)$; cf. Theorem 7.1 and D 8.2. Suppose that *not* $Q(P)<P$; by Theorem 9.2 there exists $Y\in\alpha(Q(P))-\alpha(P)$, and Y does not intersect P. By Theorem 14.7 P and Y can be separated by S-neighborhoods U and V. Since $\overline{U}$ or $\overline{U}-a(P)$ is an element of $F(P)$, and $F(P)\in Q(P)$, $F(P)$ intersects no point element of which V is an S-neighborhood. By Theorem 8.1 Y and $Q(P)$ do not intersect. Thus, the supposition that "*not* $Q(P)<P$" involves a contradiction. Since, also, $P<Q(P)$, $P=Q(P)$.

THEOREM 14.9. *If D is an S-neighborhood of the regular amalgamation point P, then the set $(S-D)\cdot a(P)$ is finite or vacuous.*

Proof. Suppose that $M=(S-D)\cdot a(P)$ is infinite. By Theorems 10.3, 10.4, and 12.3 there exists a composition point β which is an end and a limit point of M; since $S-D\supset M$, β is an end of $S-D$ (cf. D 10.2). By Theorems 10.2 and 11.3 P and β do not intersect. By Theorem 14.7 P and β can be separated by S-neighborhoods U and V. Since β is a limit point of M, V contains points of M and of $a(P)$; by Theorem 12.4 $\overline{U}\supset a(P)$. Thus, the supposition that M is infinite involves a contradiction.

THEOREM 14.10. (1) *A perfectly compact set of regular amalgamation points is amalgamable.* (2) *An amalgamation of a collection of regular amalgamation points is regular.*

Proof. By Theorem 9.1, if N is the set of (1), then N has a summation P; $N<P<\alpha(P)<N$. If the composition point X intersects no element of $\alpha(P)$, by D 9.2 X intersects no element of N. By Theorem 14.7 each element of N can be separated from X by S-neighborhoods. There exists a finite collection H of open sets which covers N relative to S, and an S-neighborhood of X, U, such that U has no point in common with an element of H (cf. the proof of Theorem IX, p. 89, Alexandroff and Hopf). Let W be the sum of the elements of H; it is an S-neighborhood of N and of $\alpha(P)$ (cf. Theorem 11.6). Thus, X is not a limit point of $\alpha(P)$. By Theorem 14.1 P is an amalgamation point.

Let the elements of N be regular, and let D be an S-neighborhood of P. If $X\in N$, D is an S-neighborhood of X and $a(P)\supset a(X)$ (cf. Theorems 11.6 and 12.4). By our hypothesis there exists an S-neighborhood of X, $U(X)$, such that $D+a(X)\supset \overline{U(X)}$. By Theorems 14.3 and 11.6 there exists a finite collection of the $U(X)$'s, whose sum U is an S-neighborhood of N and of P. Then $D+a(P)\supset \overline{U}$.

THEOREM 14.11. (1) *If a collection H of amalgamation points has a lower bound, its greatest lower bound is an amalgamation point*; (2) *if the elements of H are regular and one of them is the amalgamation of a set of regular composition points, then their greatest lower bound is regular.*

Theorems 14.5, 14.10, and 14.11 are applied in Chapter III in questions concerned with the existence and the properties of multiplicative systems and complete lattices of regular amalgamation points. Examples may be given to show that the conclusion of (2) does not hold if the second part of the hypothesis be omitted.

Proof. Let F be the product of all the $\alpha(h)$'s, where $h\in H$. If P is the summation of the elements of F, by Theorem 9.2 P is the greatest lower bound of the h's. Let the composition point Q be a limit point of F which does not intersect P. Suppose that no element of $\alpha(Q)$ is a limit point of F. By Theorem 12.2, if $\beta\in\alpha(Q)$, there exists an S-neighborhood of β, $D(\beta)$, which is not an S-neighborhood of any element of F. By Theorems 14.2, 14.3, and 11.6 there exists an S-neighborhood of Q, D, which is the sum of a finite collection of the $D(\beta)$'s. By Theorem 11.7 D is not an S-neighborhood of any element of F. Thus, we are involved in a contradiction, and there exists $\beta\in\alpha(Q)$ such that β is a limit point of F. If $h\in H$, $\alpha(h)\supset F$, and β is a limit point of $\alpha(h)$; by Theorem 14.1 β intersects an element of $\alpha(h)$. Since β is atomic, $\beta\in\alpha(h)$. Thus, $\beta\in F$; by Theorem 8.1 P and Q intersect. By Theorem 14.1 P is an amalgamation point.

We shall first establish (2) for the case where the number of elements of H is the positive integer n. Clearly (2) holds for $n=1$. Suppose that (2) holds for each collection with fewer than n elements. Let A be an element of H which is the amalgamation of a collection T of regular amalgamation points; Let $B \in H-A$, and $H'=H-B$. By the definition of n, the elements of H' have a greatest lower bound C, which is a regular amalgamation point. Let Z be the greatest lower bound of B and C. By Theorem 9.2 Z is the greatest lower bound of the elements of H; by part (1) Z is an amalgamation point. Let D be an S-neighborhood of Z. If X is an amalgamation point, let $D(X)$ denote the aggregate of all elements of $\alpha(X)$ of which D is *not* an S-neighborhood. By Theorem 14.4 $\alpha(X)$ is perfectly compact in itself. Since $D(X)$ is closed relative to $\alpha(X)$, it is perfectly compact in itself (cf. Fréchet, (I), pp. 229–230). By Theorem 11.6 D is an S-neighborhood of $\alpha(Z)$; by Theorem 9.2 $\alpha(Z)=\alpha(B)\cdot\alpha(C)$; it follows that $D(B)\cdot D(C)$ is vacuous. If $X \in D(B)$, by Theorem 8.1 X does not intersect C. By the definition of n, C is regular; by Theorem 14.7 X and C can be separated by S-neighborhoods. By Theorem 11.6 each element of $D(B)$ can be separated from each element of $D(C)$ by S-neighborhoods. Since each of $D(B)$ and $D(C)$ is perfectly compact in itself, they can be separated by S-neighborhoods D_1 and D_2 (apply an argument similar to that for the proof of Theorem IX, p. 89, Alexandroff and Hopf). Then $D+D_1$ and $D+D_2$ are S-neighborhoods, respectively, of $\alpha(B)$ and of B, and of $\alpha(C)$ and of C, respectively. Since B and C are regular, there exist S-neighborhoods R_1 and R_2 of B and C, respectively, such that $E_1=D+D_1+a(B)\supset\overline{R}_1$, and $E_2=D+D_2+a(C)\supset\overline{R}_2$. Since $Z<B$ and $Z<C$, by Theorem 12.1 $R=R_1\cdot R_2$ is an S-neighborhood of Z. Let $M_1=D+D_1$, and $N_1=(S-M_1)\cdot a(B)$; thus, $M_1+N_1=E_1$. Let $M_2=D+D_2$, $N_2=(S-M_2)\cdot a(C)$, and $K=M_1\cdot N_2+M_2\cdot N_1+N_1\cdot N_2$. By Theorem 14.9 N_1, N_2, and K are finite or vacuous. Since $D_1\cdot D_2$ is vacuous, $M_1\cdot M_2=D$, and $E_1\cdot E_2=(M_1+N_1)\cdot(M_2+N_2)=D+K$. Thus, $D+K\supset\overline{R}$. By Theorem 12.4 $\overline{R}\supset a(Z)$ and $a(B)\cdot a(C)\supset a(Z)$. Let $L=K-K\cdot a(Z)$; then L is finite or vacuous. Suppose that there exists $X\in\alpha(Z)$ and $Y\in L$ such that X intersects Y. Since $X<Z<A$, $X\in\alpha(A)$. Since $A<T$, by D 9.2 X intersects an element of T, say X_1. Let the point elements E and E_1 be elements of X and of X_1, respectively. By Theorem 8.1 E and E_1 intersect; by Theorem 5.1 $a(E)=a(E_1)$; by D 12.2 $a(X)=a(X_1)$. Since $X<Z$, by Theorem 12.4 $a(Z)\supset a(X)$; thus, $a(X_1)\neq Y$. By D 4.1 there exists $e_1\in E_1$ such that e_1 does not contain Y. Then $S-Y$ is an S-neighborhood of E_1 and of X_1. Since X_1 is regular, there exists an S-neighborhood of X_1, U, such that $S-Y\supset\overline{U}$; then $S-\overline{U}$ is an S-neighborhood of Y. Since X is atomic and intersects X_1, $X<X_1$; then U is an S-neighborhood of X. By Theorem 11.5 X does not intersect Y; thus the supposition that X and Y exist involves a contradiction. By Theorem 14.6 Z and L can be separated by a pair of their respective S-neighborhoods, U and V. Then $W=R\cdot U$ is an S-neighborhood of Z, $\overline{R}\supset\overline{W}$, and $\overline{W}\cdot L$ is vacu-

ous; since $D+K\supset\overline{R}$, $D+a(Z)\supset D+(K-L)\supset\overline{W}$. Thus, Z is regular. Thus, by induction we establish the conclusion of (2) for any case where H is finite.

Consider the infinite case. Let Z be the lower bound of the elements of H; let A, D, and $D(X)$ be defined as in the preceding paragraph. By Theorem 9.2 $\alpha(Z)$ is the product of the sets $\alpha(h)$, where $h\in H$; by Theorems 9.1 and 11.6 D is an S-neighborhood of $\alpha(Z)$; hence, the sets $D(h)$ have a vacuous product. We showed in the preceding paragraph that each of the $D(h)$'s is perfectly compact in itself; it follows that there exists a finite subcollection of H, $H'=(h_1, h_2, \cdots, h_n)$, such that the product $\lambda=D(h_1)\cdot D(h_2)\cdots D(h_n)$ is vacuous (cf. Fréchet, (I), p. 231). We shall suppose that $A\in H'$; this is permissible, for if the product λ is vacuous, the product of all the elements of λ by $D(A)$ will also be vacuous. Let J be the product $\alpha(h_1)\cdot\alpha(h_2)\cdot\alpha(h_3)\cdots\alpha(h_n)$. Let $B=C=P(J)$; by the results of the preceding paragraph B is the greatest lower bound of the elements of H', and it is a regular amalgamation point. Clearly D is an S-neighborhood of each element of J and of J; by Theorem 11.6 D is an S-neighborhood of B. Let R be an S-neighborhood of B such that $D+a(B)\supset\overline{R}$; let $K=(S-D)\cdot a(B)$ and $L=K-K\cdot a(Z)$. Then $D+K\supset\overline{R}$. By following the argument of the later portion of the preceding paragraph, we see that Z is a regular amalgamation point.

THEOREM 14.12. (1) *A Hausdorff space is normal if and only if all its atomic boundary points are regular*; (2) *it is semi-completely normal if and only if all its atomic ideal points are regular.*

Proof. By Theorems 13.2 and 13.3 the conditions are necessary. By Theorem 14.2 each composition point is an amalgamation point. If the conditions hold it follows from Theorems 14.10, 14.2, and 9.1 that each of the composition points involved in the conditions of Theorems 13.2 and 13.3 is regular. Thus, the conditions are sufficient.

THEOREM 14.13. *All the amalgamation points of a semi-completely normal space are regular, and each of them can be decomposed into a collection of regular atomic composition points.*

This follows from Theorems 9.1, 14.12 and 14.10.

EXAMPLES. E 14.3. Urysohn(16) has constructed a countable space R such that if X and Y are points of R and U and V are open sets containing X and Y, respectively, then $\overline{U}\cdot\overline{V}$ is nonvacuous; because of this condition no point of R is isolated. Let P be an amalgamation point which is distinct from the maximal amalgamation point of R; cf. D 9.4 and Theorem 14.5. Suppose that P is regular. By D 9.2 and Theorem 8.1 there exists a composition point Q which does not intersect P. By Theorems 14.2 and 14.7 P and Q can be separated by a pair of their respective R-neighborhoods, U and V. There exists an

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(16) Loc. cit., pp. 274–283.

R-neighborhood of P, W, such that $U+a(P)\supset\overline{W}$. By Theorem 14.9 there exist at most a finite number of points in $(R-U)\cdot a(P)$, say $P_1, P_2, \cdots, P_n$. Since no point of R is isolated it follows from Theorems 12.3, 11.3, and 11.6 that each of P and Q is a limit point of R. Thus, there exist $X\in V$ and $Y\in W$. For each i let δ_i and λ_i be mutually exclusive open sets which contain P_i and X respectively. Let $\lambda=V\cdot\lambda_1\cdot\lambda_2\cdots\lambda_n$. Then $\overline{V}\supset\bar{\lambda}$, $U\cdot\overline{V}$ is vacuous, $a(P)\cdot\bar{\lambda}$ and $\overline{W}\cdot\bar{\lambda}$ are vacuous; we are involved in a contradicion. *To summarize*: the maximal ideal point is a regular amalgamation point of R and is the only one; by Theorem 14.2 no composition point of R is regular.

15. **The upper semi-continuity and the perfect compactness of decompositions.** First we shall recall definitions by Moore and by Alexandroff(17), and then we shall introduce extensions of these. (I) Let T be a space *H* Fréchet, K be a point set of T, and M be a collection of mutually exclusive subsets of K whose sum is K; M is called a *decomposition* (*Zerlegung*) of K (cf. D 9.3). (II) A collection M of mutually exclusive closed point sets of T is said to be *upper semi-continuous* (*in* T) provided that if $P\in M$ and D is an open set of T which contains P, there exists in T an open set R which contains P such that if an element of M has a point in common with R then that element is a subset of D(18). (III) If $T\supset K$ and M is a decomposition of K, then $T(M)$, the (*weak*) *space of the decomposition*, is defined as follows: (1) its points are the elements of M; (2) if U is an open point set in T, let $U(M)$ be the aggregate of all elements of M which are subsets of U; the $U(M)$'s are the *neighborhoods* for $T(M)$(19) (cf. Theorem 12.1). (IV) A continuous mapping of the space X on the space Y is said to be *closed* if the image of each closed point set in X is a closed point set in Y(20).

EXAMPLES. E 15.1. Let X be the subspace of the plane whose points are those of a circle and its interior. Let δ be a definite diameter of the circle; Let Y_1 be the decomposition of X which consists of δ and all chords of the circle which are parallel to δ; let Z_1 be the decomposition whose elements are the points of δ, and the chords that are parallel to δ. Let Y and Z be the spaces of the decompositions Y_1 and Z_1 respectively. Let the relations $y=f(x)$ and $z=g(x)$ mean, respectively, that $x\in y$ and $x\in z$. The collection Y_1 is upper semi-continuous in X, but Z_1 is not. The spaces X and Y are perfectly compact, but Z is not. The mapping $y=f(x)$ of X on Y is closed. The mapping

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(17) Cf. Moore, (III), and Alexandroff, (I). For treatments of the theory see Moore, (1), chap. 5, and Alexandroff and Hopf, pp. 61–70 and 95–98.

(18) Moore, loc. cit.; Alexandroff and Hopf call M a *continuous decomposition* (*stetige Zerlegung*) of the sum of the elements of M (loc. cit., p. 67).

(19) Alexandroff and Hopf, p. 66, call $T(M)$ *der schwache Zerlegungsraum der Zerlegung* M. If M is upper semi-continuous, $T(M)$ is their *Zerlegungsraum* (loc. cit., pp. 63 and 67). We deal mainly with the latter.

(20) Alexandroff and Hopf, p. 95; for information and definitions having to do with mappings (*Abbildungen*) see pages 51, 52; in V, page 54, they show that a continuous mapping need not be closed.

$z=g(x)$ of X on Z is not continuous. The mapping $y=f(g^{-1}(z))$ of Z on Y is continuous but is not closed. The example both serves to illustrate the results of Theorem 15.1 and to give *Gegenbeispiele.* If we amalgamate the elements of Y_1 and Z_1, we obtain collections of amalgamation points which may be used to illustrate Theorem 15.4.

The extensive results of Alexandroff and Hopf in their Theorem VIII, p. 98, may be amplified by the following theorem. The mapping of their hypothesis is continuous rather than closed. They give no result analogous to that of the Conditions (I) and (III) implying (II); the transformation $y=f(g^{-1}(z))$ of the preceding example illustrates the fact that the stronger condition, *closedness*, is essential for such a result.

THEOREM 15.1. *Let X and Y be Hausdorff spaces, and let $y=\alpha(x)$ define a continuous mapping of X into Y; then any two of the following conditions imply the third*: (I) *Y is perfectly compact*; (II) *X is perfectly compact*; (III) *the mapping $y=\alpha(x)$ is closed*; *for $y\in Y$ the set $\alpha^{-1}(y)$ is perfectly compact in itself.*

Proof. By Alexandroff and Hopf, Theorem VIII, p. 98, Conditions (II) and (III) imply (I). By this theorem and their Theorems II on page 95, (I) on page 53, and IV on page 86 Conditions (I) and (II) imply (III). Let H be a collection of open sets which covers X. If $y\in Y$ let $H(y)$ be a finite subcollection of H which covers $\alpha^{-1}(y)$ (cf. Condition (III) and Kuratowski and Sierpiński, loc. cit.). Let $D(y)$ be the sum of the elements of $H(y)$; then $X-D(Y)$ is closed in X. By (III) $\alpha(X-D(y))$ is closed in Y; and $\delta(y)=Y-\alpha(X-D(y))$ is open in Y and contains y. By (I) there exists a finite subcollection of the $\delta(y)$'s which covers Y, say $\delta(y_1)$, $\delta(y_2)$, $\cdots$, $\delta(y_n)$. Let $H^*=H(y_1)+H(y_2)+\cdots+H(y_n)$. If $x_0\in X$, there exists j such that $\alpha(x_0)\in\delta(y_j)$; if x_0 were not covered by $D(y_j)$, it would belong to $X-D(y_j)$; then $\alpha(x_0)\in\alpha(X-D(y_j))=Y-\delta(y_j)$, and we are involved in a contradiction. Thus x_0 is covered by an element of $H(y_j)$, and X is covered by H^*. Thus, (I) and (III) imply (II).

D 15.1. A collection M of points is said to be *upper semi-continuous* (*relative to* S) provided that if $P\in M$ and D is an S-neighborhood of P, there exists an S-neighborhood of P, R, such that if Q is an element of M and R is an S-neighborhood of a portion of Q, then D is an S-neighborhood of Q (cf. Definition (II) above).

EXAMPLES. E 15.2. Let M be a decomposition of the perfectly compact Hausdorff space S into closed sets. Let N be the collection of those ideal points which are amalgamations of elements of M. Then N is upper semi-continuous relative to S if and only if M is upper semi-continuous in S. Examples of both possibilities are given by Alexandroff and Hopf, p. 67. See also E 15.1.

E 15.3. A collection of real points, or of atomic ideal points, is upper semi-continuous relative to S.

If, in Theorem 15.1, we amalgamate the elements of the sets $\alpha^{-1}(y)$, for

$y \in Y$, we obtain amalgamation points for which Theorem 15.4 applies. This is an instance of the analogy we have noted before between the properties of perfectly compact sets and of amalgamation points. Similarly, the results of Theorem 15.3 resemble some of those of Alexandroff and Hopf. These analogies give indications of some rather trivial applications of our theory. In general, our procedure is to define mappings in terms of decompositions or of amalgamations; these processes involve mainly order relations. This procedure must be carried out in such a manner that it gives conditions which assure the continuity of the mappings and permit the application of the results of Alexandroff and Hopf. Such conditions are given in Theorems 15.4, 15.5, and 15.6. Thus, we have on the one hand the processes of amalgamation of points and of continuous mappings, and on the other those of decomposition of points and of the inverse of a continuous mapping. We may think of the latter as giving a kind of representation theory(21) for our system of points, in which the representation of a point is a decomposition of that point into an upper semi-continuous collection of amalgamation points. Thus, this representation theory involves not only order, but also continuity.

THEOREM 15.2. *If N is an upper semi-continuous collection of amalgamation points, no two of them intersect.*

Suppose that X and Y are intersecting elements of N. By D 15.1 X and Y have the same S-neighborhoods. By Theorem 9.2 if not $X < Y$ then there exists an element β of $\alpha(X) - \alpha(X) \cdot \alpha(Y)$. By Theorem 12.2 if $\lambda \in \alpha(Y)$ there exists an S-neighborhood of λ, $\delta(\lambda)$, which is not an S-neighborhood of β. By Theorem 14.3 there exists a finite collection of the $\delta(\lambda)$'s, whose sum D is an S-neighborhood of Y. By Theorem 11.7 D is not an S-neighborhood of β. Since $\beta < X$ and D is an S-neighborhood of X, we are involved in a contradiction. Thus, $X < Y$. Similarly, $Y < X$. Hence, $Y = X$.

THEOREM 15.3. *Let Y be a decomposition*(22) $[x]$ *of the ideal point X into regular amalgamation points. The following conditions are equivalent*: (1) *Y is perfectly compact in itself*; (2) *X is an amalgamation point and Y is upper semi-continuous relative to S*; (3) *the space*(23) *$S(Y)$ of the decomposition Y is a perfectly compact Hausdorff space.*

The following is a point set analogue of this theorem.

THEOREM 15.3A. *If Y is a decomposition of the Hausdorff space X into a collection of sets which are perfectly compact in themselves, the following conditions are equivalent*: (1) *The space $X(Y)$ of the decomposition Y is perfectly compact*; (2) *X is perfectly compact and Y is upper semi-continuous in X.*

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(21) Cf. Birkhoff, p. 76, and Stone, loc. cit.

(22) Cf. D 9.3.

(23) Cf. Theorem 12.1.

Consider 15.3A. By Alexandroff and Hopf, p. 98, Theorem VIII, (2) implies (1). Let (1) hold and $y=\alpha(x)$ mean that $x\in y\in Y$ $(X\supset y)$. If U is an open set in X, $U(Y)$ means the set of all elements of Y that are subsets of U; and $U(Y)$ is an open set in $X(Y)$ (cf. Definition (III)). Let $y\in Y$, and R be an open set in X containing y. If $z\in Y-R(Y)$, there exist in X mutually exclusive open sets, V_z and W_z, which contain, respectively, y and z (cf. proof, Theorem IX, p. 89, Alexandroff and Hopf). Since $Y-R(Y)$ is a closed set in the perfectly compact space $X(Y)$, it has the Borel-Lebesgue property; there exist $W_{z_1}(Y)$, $W_{z_2}(Y)$, $\cdots$, $W_{z_n}(Y)$ which cover $Y-R(Y)$ in $X(Y)$. Let $V=V_{z_1}\cdot V_{z_2}\cdots V_{z_n}$ and $W=W_{z_1}+W_{z_2}+\cdots+W_{z_n}$. Then V and W are mutually exclusive open sets in X, $V\supset y$, and W contains all elements of $Y-R(Y)$. Since $V\cdot W$ is vacuous, any element of Y that has a point in common with V is a subset of R. Thus, Y is upper semi-continuous in X; and the mapping $y=\alpha(x)$ of X on $X(Y)$ is continuous (cf. Alexandroff and Hopf, p. 67). Let F be a closed point set in X, and $D=X-F$. Then $D(Y)$ is open in $X(Y)$. Since $D(Y)=X(Y)-\alpha(F)$, $\alpha(F)$ is closed in $X(Y)$, and the mapping is closed. By Theorem 15.1 Condition (1) implies (2).

Proof of Theorem 15.3. Let (2) hold and G be a collection of open sets of S that covers Y relative to S. If $x\in Y$, let $D\in G$ such that D is an S-neighborhood of x. Let $R(x, D)$ be an S-neighborhood of x such that if it is an S-neighborhood of a portion of an element of Y, then D is an S-neighborhood of that element. By Theorems 14.4, 11.6, and 9.1 there exists a finite collection of the R's which covers the aggregate of atomic portions of X; say $R(x_1, D_1)$, $R(x_2, D_2)$, $\cdots$, $R(x_n, D_n)$. Let G^* be the collection $D_1, D_2, \cdots, D_n$. If $x\in Y$ and β is an atomic portion of x, $\beta<x<X$, and there exists j such that $R(x_j, D_j)$ is an S-neighborhood of β. Then D_j is an S-neighborhood of x, and G^* covers Y relative to S. Since Y has the Borel-Lebesgue covering property, it is perfectly compact in itself (cf. Theorem 12.1 and Kuratowski and Sierpiński, loc. cit.). Thus (2) implies (1).

Conversely, let Y be perfectly compact in itself. By Theorem 14.10 X is an amalgamation point. Let $x\in Y$, D be an S-neighborhood of x, and L be the set of all elements of Y of which D is not an S-neighborhood. By D 12.1 no point of $Y-L$ is a limit point of L, and L is a closed point set relative to Y. Since Y is perfectly compact in itself, so is L. By Theorem 14.10 there exists a regular amalgamation point β, such that $\beta<L<\beta$. By D 9.3 x intersects no element of L; and thus it does not intersect β. By Theorem 14.7 x and β can be separated by a pair of their respective S-neighborhoods, U and W; by Theorem 11.6 W is an S-neighborhood of L. Let x_1 be an element of Y such that U is an S-neighborhood of a portion λ of x_1. If x_1 were an element of L, W would be an S-neighborhood of x_1 and of λ; since $U\cdot W$ is vacuous, this involves a contradiction. Since $x_1\in Y-L$, D is an S-neighborhood of x_1. Thus, Y is upper semi-continuous, and (1) implies (2).

If x_1 and x_2 are two elements of Y, by Theorem 14.7 and D 9.3 they can be separated by S-neighborhoods. By Theorem 12.1 condition (1) implies (3). The converse is obvious.

THEOREM 15.4. *Given that P is an amalgamation point and Y and X are decompositions of P into regular amalgamation points such that if $y \in Y$ there exists a subcollection $\alpha^{-1}(y)$ of X which is a decomposition of y; then any two of the following conditions imply the third*: (1) *Y is perfectly compact in itself*; (2) *X is perfectly compact in itself*; (3) *the mapping $y=\alpha(x)$ of X into Y is a closed mapping; the sets $\alpha^{-1}(y)$ for $y \in Y$, are perfectly compact in themselves.*

By Theorem 15.3 the conditions (2) and (1) are equivalent, respectively, to the upper semi-continuity of X and of Y. Cf. E 15.1.

The close resemblance of Theorems 15.1 and 15.4 fails to extend to the explicit assumption of the continuity of the mapping $y=\alpha(x)$, which is necessary for the former. The inherent potentialities for the continuity of $y=\alpha(x)$ for Theorem 15.4 are an example of the properties of this mapping which follow from the way it is defined: For $y \in Y$, $y<\alpha^{-1}(y)<y$; $X \supset \alpha^{-1}(y)$; and if $x \in \alpha^{-1}(y)$, then $x<y$ (cf. D 9.3). These order relations, which involve X, Y, and the mapping, may be regarded as a key to our theory, and are a foundation for the applications we make in following sections. The transformation expresses the fact that the elements of y are obtained by amalgamations of elements of X; conversely, the elements of X are the results of decompositions of the y's. For topological applications we need conditions of continuity as well as those for order; simultaneous conditions for the required continuity and order are given in Theorems 15.3 to 15.6.

Proof. By Theorem 15.3 if condition (1) or (2) holds, then Y or X, respectively, is upper semi-continuous relative to S. By (1) and (2) the mapping $y=\alpha(x)$ is continuous from X to Y (cf. argument on page 67, and Theorem IV, p. 53, Alexandroff and Hopf). The conclusion follows from Theorem 15.1.

THEOREM 15.5. *Adopt the notation of Theorem* 15.4, *and let X be upper semi-continuous*(24) *relative to S; then* (A) *Y is upper semi-continuous relative to S if and only if the aggregate $Y'=[\alpha^{-1}(y)]$, where $y \in Y$, is an upper semi-continuous collection of point sets in X*(25). (B) *Let $Y''=X(Y')$ be the space*(26) *of the decomposition of X, Y', and let $y''=\alpha''(x)$ mean that $x \in y'' \in Y''$; then the mapping $y=\alpha(x)$ of X on Y is closed if and only if the mapping $y''=\alpha''(x)$ of X on Y'' is closed.* (C) *If the conditions in either* (A) *or* (B) *are satisfied, then those in each of Theorem* 15.4, (A) *and* (B), *are satisfied, and the mappings $y=\alpha(x)$ and $y''=\alpha''(x)$ are equivalent mappings of X*(27).

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(24) Or, X is perfectly compact in itself (cf. Theorem 15.3).

(25) Cf. definition (II). Here X is regarded as a space (cf. Theorem 12.1).

(26) Cf. Definition (III); Y'' is *der zur Zerlegung gehörende schwache Zerlegungsraum* (cf. Alexandroff and Hopf, p. 66). The *points* of Y'' are the *elements* of Y'.

(27) That is, the mapping, $y''=\alpha''(\alpha^{-1}(y))$, of Y on Y'' is a homeomorphism (cf. Alexandroff and Hopf, p. 61).

Proof. By Theorems 14.7 and 12.1 X and Y determine Hausdorff spaces. Let two of the conditions of Theorem 15.4 hold. The third follows; by Theorems VIII and II, pages 98 and 95, Alexandroff and Hopf, the mappings of (B) are equivalent and both are closed, and Y' is upper semi-continuous in X.

By Theorem 15.3 our hypothesis that X is upper semi-continuous relative to S is equivalent to condition (2) of Theorem 15.4. If $y=\alpha(x)$ is a closed mapping, condition (3) of that theorem holds. By the preceding paragraph the conditions in (A) and in (B) are necessary. Let $y''=\alpha''(x)$ be a closed mapping from X to Y''. Let $y\in Y$, U be an S-neighborhood of y, and $U(X)$ and $U(Y)$ denote all elements of X and of Y, respectively of which U is an S-neighborhood; $U(X)$ is open relative to X, and $U(X)\supset\alpha^{-1}(y)=y''$. If V is the set of all elements of Y'' that are subsets of $U(X)$, then V is an open set in Y'', and $y''\in V$ (Definition (III)). Let H be the sum of the elements of $Y''-V$. Then y'' and H are closed point sets in X which do not intersect (cf. Theorem I, p. 53, Alexandroff and Hopf). Since X is perfectly compact, so are y'' and H. By Theorems 14.10, 14.7, and 11.6 y'' and H can be separated by S-neighborhoods D and R. Since $y''=\alpha^{-1}(y)$, $y<y''<y$; by Theorem 11.6 D is an S-neighborhood of y. Let D be an S-neighborhood of a portion β of an element of Y, y_1. Suppose that $y_1\in Y-U(Y)$; by Theorem 11.6 there exists $x_1\in\alpha^{-1}(y_1)$ such that U is not an S-neighborhood of x_1. Then $\alpha^{-1}(y_1)=\alpha''(x_1)\in Y''-V$, and $H\supset\alpha^{-1}(y_1)$. By Theorem 11.6 R is an S-neighborhood of y_1 and of β. Since D is an S-neighborhood of β, and $R\cdot D$ is vacuous, we are involved in a contradiction. Thus, Y is upper semi-continuous relative to S. By Theorems 15.3 and 15.4 the condition in (B) is sufficient. If the collection Y' is upper semi-continuous in X, it follows from Theorems VIII and II on pages 98 and 95, Alexandroff and Hopf, that the mapping $y''=\alpha''(x)$ is closed from X to Y''. By (B) the mapping $y=\alpha(x)$ is closed. By the first paragraph the condition in (A) is sufficient, and (C) is true.

THEOREM 15.6. *The conclusions of Theorems* 15.3, 15.4, *and* 15.5 *remain true if the aggregate consisting of P and of the elements of X and Y include regular real or regular frontier points.*

This follows from Theorems 12.1, 13.7, and 14.2.

16. **Applications of the preceding sections.** Applications of the preceding section require the decomposition of an amalgamation point P into regular amalgamation points. In problems dealing with the embedding of S in a perfectly compact Hausdorff space we have the case where P is the maximal amalgamation point and the elements of the decomposition include the points of S (cf. Theorem 16.1).

We supplement the results of §§13 and 14 for the regular, the normal, and the semi-completely normal spaces by giving characterizations of the completely regular, the locally perfectly compact, and the perfectly compact

Hausdorff spaces. Some of these give our interpretations of results by other authors, and demonstrate the generality of our methods (cf. Theorems 16.1, 16.2, and 16.4). In order to utilize this generality fully, we devote Chapter III to a study of lattices of regular points, and to lattices of systems of decompositions of an amalgamation point into collections of regular amalgamation points. These give systematic methods for classifying our results. An important part of Chapter III is the demonstration of the existence of atomic elements of systems of points which are encountered in applications of our mapping theory. For the special case of the semi-completely normal space the required atomic elements are the atomic ideal points (cf. Theorem 16.5). Similar conclusions hold for the decompositions of *boundary* points of *normal* spaces (cf. Theorem 16.1, (3)).

THEOREM 16.1. (1) *In order that S be completely regular*(28), *it is necessary and sufficient that there exist a collection of regular boundary points M such that* (a) *each boundary point intersect M, and* (b) *the collection $S+M$ be upper semi-continuous relative to S.* (2) *This condition is satisfied if and only if $S+M$ is a perfectly compact Hausdorff space.* (3) *If S is normal and M is the set of its atomic boundary points, this condition is satisfied.* (4) *A perfectly compact, immediate Hausdorff extension of S is a decomposition of the maximal ideal point.*

The result in (3) is similar to that in Lemma 12, p. 119, Wallman, loc. cit. If Y is any perfectly compact Hausdorff space in which S is embedded and X is the space $S+M$ of (3), then the conclusion of Theorem 15.4 concerning the mapping $y=\alpha(x)$ holds; cf. Stone, loc. cit., p. 476, Theorem 88. See also Theorem 20.2. Čech, also, loc. cit., has considered this space.

Proof. Let Y be the maximal amalgamation point and $X=S+M$. Let P be a real point, Q be a boundary point, and β be an atomic portion of Q. By (1a) β intersects $\beta_M \in M$; since β is atomic, $\beta<\beta_M$. Since β_M is regular, P and β_M do not intersect, and they can be separated by S-neighborhoods U and V (cf. D 13.1 and Theorem 11.5). Since $\beta<\beta_M$, V is an S-neighborhood of β and P and β do not intersect (Theorems 11.6 and 11.5). By Theorem 14.2 Q is an amalgamation point. By Theorem 14.6 P and Q do not intersect. By Theorem 13.1 P is regular. By (1a) each composition point intersects either S or M. By D 9.2 $Y<M+S$; since $M+S<Y$, $X=S+M$ is a decomposition of Y (cf. Theorem 15.2). By Theorems 14.5, 15.6, and 15.3 the condition in (1) is sufficient and that of (2) is necessary.

Conversely, let the perfectly compact Hausdorff space T be an immediate extension of S; then all points of T are regular. Let M be the boundary points of S which are equivalent to points of $T-S$. By Theorems 13.7 and 12.1,

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(28) S is completely regular if and only if it can be embedded in a perfectly compact Hausdorff space; cf. Tychonoff, loc. cit.

$S+M$ is perfectly compact. By Theorems 14.10 and 14.2 there exists an amalgamation Z of the elements of $S+M$. If $Z\neq Y$, there exists an atomic boundary point β which intersects Y but does not intersect Z (cf. Theorem 9.2). By Theorems 12.1 and 14.7 the space $S+M+\beta$ is a Hausdorff space; clearly, β is a limit point of S. Since $S+M$ is perfectly compact, we are involved in a contradiction; cf. Alexandroff and Hopf, p. 91, Theorem XI. Thus, $Y<Z$, $Z<Y$, and $Z=Y$. By Theorems 15.6 and 15.3 the condition in (1) is necessary, and that of (2) is sufficient.

For Case (3) the collection $X=S+M$ obviously is upper semi-continuous. By Theorems 5.5 and 14.12 M satisfies condition (1a).

THEOREM 16.2. (1) *If S is regular but is not perfectly compact, then in order that the set of all boundary points be amalgamable it is necessary and sufficient that S be locally perfectly compact*; (2) *if this condition is satisfied and Q is the amalgamation of all the boundary points, then S+Q is perfectly compact.*

This theorem resembles closely one due to Alexandroff; cf. Alexandroff and Hopf, p. 93, Theorem XIV.

Proof. If the condition holds, it follows from Alexandroff's theorem that there exists a point Q, such that $S+Q$ is a perfectly compact Hausdorff space in which S is embedded; because of Theorems 13.7 and 12.1 we may suppose that Q is a boundary point; by Theorem 14.2 Q is an amalgamation point. It follows from Theorem 16.1 that each boundary point is a portion of Q. Thus, the condition is sufficient.

Conversely, let Q be the amalgamation of all the boundary points. By Theorems 13.1 and 14.6 Q and the real point P can be separated by S-neighborhoods U and W, respectively. If $\overline{W}$ were not perfectly compact, by Theorem 10.3 there would exist a boundary point β which is an end of $\overline{W}$. Then β is a portion of Q, and U is an S-neighborhood of β. By Theorem 12.3 β is a limit point of $\overline{W}$, and $W\cdot U$ is nonvacuous. Thus, we are involved in a contradiction, and the condition is necessary.

THEOREM 16.3. *In order that a Hausdorff space be perfectly compact, it is necessary and sufficient that it have no boundary points.*

This follows from Theorem 10.3.

THEOREM 16.4. *Let X be a perfectly compact Hausdorff space which is an immediate extension of S, P be the maximal S-portion, M be a decomposition of X into closed point sets, and Y be the aggregate of all ideal points,* $y(m)$, *where* $y(m)$ *is the amalgamation of the elements of* m, *and* $m\in M$. *Theorems* 15.4 *and* 15.5 *are applicable.*

Thus, Y is upper semi-continuous relative to S if and only if M is upper semi-continuous in X; etc. The theorem points out that the results of Alexandroff and Hopf are special cases of our's.

THEOREM 16.5. *If S is semi-completely normal, the results of Theorems* 15.4 *and* 15.5 *hold true for the case that X is the decomposition of P into atomic ideal points.*

An interesting case is the one where P is the maximal ideal point, and Y is a perfectly compact Hausdorff space in which S is embedded. This justifies our regarding X as a *universal* inverse mapping space.

17. **Extensions of upper semi-continuous collections of point sets in normal spaces.** Stone has commented on the "remarkable properties" of a space which for the case that S is normal is a homeomorph of S plus its atomic boundary points (loc. cit., p. 476, lines 8 and 9). The results of this section give additional grounds for this comment. Our results in this section are distinguished by the fact that they are *characteristic* of normal spaces. Also, cf. Čech, loc. cit.

D 17.1. If S is a subspace of T, M and N are collections of mutually exclusive closed point sets of S and of T, respectively, and M is the collection of all sets $S \cdot n$, where $n \in N$, we say that N is *an extension of M (from S into T)*.

E 17.1. Let T be a circle plus its interior, and S be the interior. Let N be the set of all chords parallel to a given diameter; define M as in D 17.1. N contains an extension of M, N is upper semi-continuous in T, but M is *not* upper semi-continuous in S. Thus, T cannot take the place of the space $\lambda(S)$ of Theorem 17.1.

Let E and F be two chords in T such that E and F have in common exactly one point, P, which belongs to $T-S$. Let K be the collection whose elements are $E \cdot S$, $F \cdot S$, and the points of $S-S\cdot(E+F)$. Then K is upper semi-continuous in S, but *cannot* be extended to T. It follows that T does not serve as a T_K (cf. Theorem 17.2).

THEOREM 17.1. *Let S be normal, $\lambda(S)$ be the space of S and its atomic boundary points, M be a collection of mutually exclusive closed point sets of S, N be the aggregate of the closures in $\lambda(S)$ of the elements of S, and K be the aggregate of the amalgamations of the elements of N.* (1) *The following conditions are equivalent*: (a) *M is upper semi-continuous in S*; (b) *N is upper semi-continuous in $\lambda(S)$*; (c) *K is upper semi-continuous relative to S.* (2) *If these conditions are satisfied, the spaces of these decompositions are homeomorphic*(29).

Thus, the theory of upper semi-continuous collections of point sets in a normal space may be regarded as a part of the theory of our ideal points (this holds true even if the elements of M are not perfectly compact).

Proof. For $m \in M$, let $n(m) \in N$ such that $m = S \cdot n(m)$; let $P(m)$ be the amalgamation of the elements of $n(m)$; by Theorems 16.1 and 14.10 $P(m)$ exists. If U is an open set in S, and $U \supset m \in M$, let $\lambda(U)$ denote all points of $\lambda(S)$ of which U is an S-neighborhood. Suppose that $\beta \in n(m) - m$ and not

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(29) Cf. Definition (III), §15, and Theorem 12.1.

$\beta \in \lambda(U)$. Then by Theorems 11.3 and 12.3 β is an end of $S-U$ and of m. By Theorem 11.3 neither U nor $S-m$ is an S-neighborhood of β. Since $S=U+(S-m)$ and S is an S-neighborhood of β, by Theorem 11.7 we are involved in a contradiction. Thus, U is an S-neighborhood of $n(m)$ and of $P(m)$; cf. Theorem 11.6. Conversely, if W is an open set in $\lambda(S)$, and $W \supset n(m)$, then $n(m)$ and $\lambda(S)-W$ are closed in $\lambda(S)$. Thus, $n(m)$ and $\lambda(S)-W$ are perfectly compact in themselves and are amalgamable (cf. Theorems 16.1 and 14.10); by Theorems 14.7 and 11.6 they can be separated by S-neighborhoods, D and R. Then $W \supset \lambda(D) \supset n(m)$, $D \supset m$, and D is an S-neighborhood of $P(m)$. By Theorem 14.10 each $P(m)$ is regular. The conclusion follows from these relations, the definitions of upper semi-continuity, and Theorem IV, p. 53, Alexandroff and Hopf.

THEOREM 17.2. (1) *In order that it be possible to extend each* M, *which is an upper semi-continuous decomposition of* S *into closed point sets, to a subset of a similar decomposition of some immediate, perfectly compact, Hausdorff extension of* S, *say* T_M, *it is necessary and sufficient that* S *be normal.* (2) (a) *If* S *is normal, there exists a* T_M, Y, *which is independent of* M; (b) *such a* Y *is homeomorphic to* $X=\lambda(S)$ *by means of the mapping* $y=\alpha(x)$ *of Theorems* 15.6 *and* 15.4.

In particular, the conclusion of (2) holds for $Y=\lambda(S)$; cf. Theorem 17.1.

Proof. Let E and F be mutually exclusive closed sets in S. Let M be the aggregate whose elements are E, F, and the points of $S-(E+F)$. Then M is an upper semi-continuous decomposition of S. If a T_M exists, there exists an upper semi-continuous extension of M into T_M. Then the product of the closures of E and F in T_M is vacuous. Since T_M is perfectly compact, it is normal (cf. Alexandroff and Hopf, p. 89). There exist in T_M mutually exclusive open sets, U and V, which contain $\overline{E}_T$ and $\overline{F}_T$, respectively. Since $U \cdot S \supset E$ and $V \cdot S \supset F$, the condition in (1) is necessary.

Conversely, let S be normal and T be the space of the decomposition M (cf. Definition (III), §15). If $t \in T$ and $s \in S$, let the relation $t=f(s)$ mean that $s \in t$ (recall that the points of T are the elements of M). Then the mapping of S on T, $t=f(s)$, is continuous and T is normal (cf. Alexandroff and Hopf, pp. 67, 53, and 70). By Theorem 16.1 there exists an immediate, perfectly compact, Hausdorff extension of T, say R. There exists a continuous mapping from $\lambda(S)$ to R, say $r=F(s)$, such that if $P \in S$, then $f(P)=F(P)$ (cf. Theorem 16.1 and Stone, loc. cit., Theorem 88, p. 476).

Let N be the aggregate $[F^{-1}(r)]$, where r ranges over R. By Theorem VIII, p. 98, Alexandroff and Hopf, N is an upper semi-continuous decomposition of $Y=\lambda(S)$ into closed point sets. Thus, we have established (2a) and the sufficiency of the condition in (1).

Consider any Y which is independent of M, and let X be the space consisting of S and those boundary points which are equivalent to points of Y; the mapping of the conclusion is a homeomorphism of X and Y (cf. Theorem

13.7). Suppose that the element δ of $X-S$ is not atomic; then there exist two atomic boundary points which are portions of it, say α and β. By Theorem 14.12 α and β are regular. By Theorem 14.7 there exist S-neighborhoods of α and of β whose closures in S, E and F are mutually exclusive. If M is the collection whose elements are E, F, and the points of $S-(E+F)$, it is an upper semi-continuous decomposition of S. Clearly, α is a limit point of E and β is a limit point of F; by Theorem 11.6 each S-neighborhood of δ contains points of E and of F, and δ is a limit point of E and of F. Hence, there exist no mutually exclusive closed sets in Y which contain E and F, respectively. Since Y contains an extension of M, we are involved in a contradiction. Thus, δ does not exist, and $X=\lambda(S)$.

CHAPTER III. THE LATTICE-MAPPING THEORY OF SYSTEMS OF REGULAR POINTS

Let P be a regular amalgamation point, and let X, Y, and $y=\alpha(x)$ be such that the conditions of Theorem 15.4 are satisfied. Recall that for $y\in Y$, $X\supset\alpha^{-1}(y)$ and y is an amalgamation of the elements of $\alpha^{-1}(y)$. Since the mapping $y=\alpha(x)$ deals both with order and with continuity, we may think of Y as a mapping space or an amalgamation space for X, and of X as an inverse mapping space or a decomposition space for Y. If S is semi-completely normal, we have shown in Theorem 16.5 that if X^* is the decomposition of P into atomic ideal points, then X^* serves as a common inverse mapping space for all the Y's which are upper semi-continuous, or perfectly compact, decompositions of P into regular amalgamation points. That is, the elements of X^* may be said to be *atomic* from the point of view of our *mapping theory*. By Theorem 14.12 for spaces other than the semi-completely normal the atomic mapping points will not, in general, be the atomic ideal points. For these more general spaces there is thus the question of the existence of such atomic elements, and that of the existence of decomposition spaces or of inverse mapping spaces.

In §19 we give conditions for which these questions have answers in the affirmative. Our methods involve an order relation $X<Y$, where X and Y are decompositions of P which satisfy the conditions of Theorem 15.4. The ordered set so obtained is a complete lattice and its zero is the required set of atomic mapping points. The sublattices of this lattice, and their zeros and units, give an extensive body of information which is not in Chapter II for the case even of the semi-completely normal space. Important sublattices are considered in §20.

Section 18 is concerned with preliminary methods dealing with multiplicative systems and lattices of ideal points, and with the generation of such systems from collections of points. The zeros of the sublattices of §19 are, in general, collections of atomic elements of systems of ideal points.

In Example 14.3 we established the existence of a space for which the only regular amalgamation point is the maximal ideal point P; for it $P=X=Y$, and the application of Theorem 15.4 is extremely limited.

18. The generation of multiplicative systems and lattices of amalgamation points.

We develop methods for generating such systems from any collection of amalgamation points. Most of these do not require a hypothesis of regularity. In Theorem 18.5 we give conditions for regularity. In Theorem 18.4 we give conditions which make the generated set a lattice.

Note that if N is a subsystem of M, an atomic element(30) of N need not be an atomic element of M (for instance, N might consist of a single non-atomic element of M). The term *atomic regular* (*amalgamation*) *point* means an atomic element of the system of all regular amalgamation points; it should not be confused with the term regular atomic point (of S). Theorems 14.2 and 14.12 show that the terms are synonymous only for the semi-completely normal spaces.

D 18.1. The set of points M is said to be *almost-multiplicative* provided that if $M\supset K$ and K has a lower bound in the system of all ideal points, then the greatest lower bound of K belongs to M. The following are examples of almost-multiplicative systems: all atomic points; all composition points; all amalgamation points; all regular composition points; if S is completely regular, all regular amalgamation points (cf. Theorems 5.4, 14.2, 14.11, and 20.1).

D 18.2. If M is almost-multiplicative but is not completely multiplicative(31), it becomes a complete multiplicative system $M+\overline{O}$ by the addition of a zero element $\overline{O}$; if M is completely multiplicative, let $M+\overline{O}$ denote M, and let $\overline{O}$ denote its zero.

These systems find applications, among others, in establishing the existence of atomic elements. If M is the system of all regular amalgamation points of the space of Example E 14.3, it has one element, the maximal ideal point of the space. This is the atomic element of the system. This example shows how far the atomic regular points may differ from the atomic points.

THEOREM 18.1. *Let M be an almost-multiplicative system of amalgamation points.* (1) *If $P\in M$, there exists an atomic element of M which is a portion of P*; (2) *no two atomic elements of M intersect*; (3) *M is completely multiplicative if and only if it has exactly one atomic element* (*its zero*).

Proof. By Hausdorff, (I), pp. 140–141, there exists a system of elements of M, say K, which contains P, is monotonic, and is not a proper subset of any monotonic subcollection of M. For $k\in K$ the set $\alpha(k)$ is perfectly compact in itself (cf. Theorems 9.1 and 14.4). By Theorem 9.2 the $\alpha(k)$'s are the elements of a monotonic collection of point sets. There exists an atomic point δ which is common to all the $\alpha(k)$'s (cf. Kuratowski and Sierpiński, and Moore,

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(30) Cf. D 3.3.

(31) Cf. D 3.7, and MacNeille, loc. cit., pp. 429, 442, and 443.

(II)). It follows that δ is a portion of each element of K. By D 18.1 there exists an element of M, β, which is the greatest lower bound of the elements of K. Let λ be an element of M such that $\lambda<\beta$, and let $k\in K$. Then $\lambda<\beta<k$. It follows that $K+\lambda$ is monotonic; because of the definition of K, $\lambda\in K$; then $\beta<\lambda$; since $\lambda<\beta$, $\lambda=\beta$, and β is an atomic element of M. Since $P\in K$, $\beta<P$.

Let α and β be intersecting atomic elements of M. By Theorem 8.1 and D 18.1 there exists an element of M, δ, which is a lower bound of α and β. Since $\delta<\beta$ and β is atomic, $\beta<\delta$, and $\beta=\delta$; similarly, $\delta=\alpha$. The conclusion of (2) and the necessity of the condition in (3) follow. If M has exactly one atomic element, it follows by (1) that this element is a lower bound of each collection of elements of M; by D 18.1 the condition in (3) is sufficient.

D 18.3. If K and $M(K)$ are collections of amalgamation points, $M(K)$ is an almost-multiplicative system, and $M(K)$ consists of those amalgamation points which are greatest lower bounds of subcollections of K, we say that *$M(K)$ is generated by K*; if $A(K)$ is the aggregate of all amalgamation points P such that P is the amalgamation of the elements of some subcollection of $M(K)$, we say that *$A(K)$ is the additive system generated by K*. If $x\in K$, x is the lower bound of the pair x and x; thus $M(K)\supset K$; similarly $A(K)\supset M(K)$. If K is either the set of all composition points, or of all atomic ideal points, then $K=M(K)$; if S is semi-completely normal, $A(K)$ is the set of all amalgamation points.

Theorem 18.2. *Each collection of amalgamation points K generates an additive system $A(K)$ and an almost-multiplicative system $M(K)$. $A(K)$ is an almost-multiplicative system, and $A(K)$ and $M(K)$ have the same atomic elements.*

Proof. Let $M(K)$ and $A(K)$ be defined as in D 18.3. Let H be a subcollection of $M(K)$ that has a lower bound. Let $H=H_1+H_2$ such that $K\supset H_1$ and $M(K)-K\supset H_2$. Let F be the product of all sets $\alpha(h)$, where $h\in H$ (cf. D 8.5). For $h\in H_2$ let $H(h)$ be a subcollection of K such that h is the greatest lower bound of the elements of $H(h)$; cf. D 18.3. Let H_3 be the sum of all the $H(h)$'s, for $h\in H_2$. Let $G=H_1+H_3$, and let E be the product of all the $\alpha(g)$'s, where $g\in G$. Clearly, $E\supset F$. Suppose that $\beta\in E-F$; there exists $h\in H$ such that β is not an element of $\alpha(h)$. Since $G\supset H_1$, $h\in H_2$; by Theorem 9.2 there exists $h_3\in H(h)$ such that $\alpha(h_3)$ does not contain β. Since $h_3\in G$, we are involved in a contradiction, and β does not exist. Thus, $E=F$; by Theorem 9.2 $P(F)$ is the greatest lower bound of the elements of H and of G. Since $K\supset G$, it follows from the definition of $M(K)$ that $P(F)\in M(K)$ (by Theorem 14.11 $P(F)$ is an amalgamation point). Thus, $M(K)$ is an almost-multiplicative system of amalgamation points, and is generated by K.

Next let H be a subcollection of $A(K)$, and let F be the product of all the sets $\alpha(h)$, for $h\in H$. If $h\in H$, there exists a subcollection of $M(K)$, M_h, such that h is the amalgamation of the elements of M_h (cf. D 18.3). For $\beta\in F$ and $h\in H$, let $x(\beta, h)$ be an element of M_h that intersects β. For a fixed β, let $x(\beta)$

be the greatest lower bound of the $x(\beta, h)$'s, where h ranges over H; by the first paragraph, $x(\beta)\in M(K)$. By Theorem 9.2 $F\supset\alpha(x(\beta))$, $x(\beta)<P(F)$, and $P(F)$ is the greatest lower bound of the elements of H. Let X be the aggregate of all the $x(\beta)$'s, for $\beta\in F$. By Theorems 8.1, 9.2, and D 9.2, $X<P(F)$, and $P(F)<X$. By Theorem 14.11 and D 14.2 $P(F)$ is an amalgamation of the elements of X. Since $M(K)\supset X$, it follows that $P(F)\in A(K)$, and that $A(K)$ is an almost-multiplicative system.

Since $A(K)\supset M(K)$, and for $a\in A(K)$ there exists $m\in M(K)$ such that $m<a$, it follows that the two systems have the same atomic elements.

Theorem 18.3. *Let K be a system of amalgamation points, and $M(K)$ be the almost-multiplicative system generated by it. Then K and $M(K)$ generate the same additive system; and, if K is almost-multiplicative, $M(K)=K$.*

Theorem 18.4. *If $A(K)$ is the additive system generated by the collection of amalgamation points K, then $A(K)+\overline{O}$ is a complete lattice if and only if it is possible to amalgamate the elements of K. If the condition is satisfied, the unit of the lattice is the amalgamation of the elements of K.*

Proof. Suppose that $A(K)+\overline{O}$ is a complete lattice; then it has a unit, I, which is an element of $A(K)$. Since $A(K)\supset K$, for $k\in K$, $k<I$; by D 9.2, $K<I$. Since $I\in A(K)$, it follows from D 18.3 and D 14.2 that there exists a subset of $M(K)$, say M, such that $I<M$. Let X be an ideal point that intersects I; by D 9.2 X intersects an element of M, say m. By the definition of $M(K)$ there exists $k\in K$ such that $m<k$. Then by D 9.2 X intersects k, and $I<K$; since $K<I$, it follows from D 14.2 that I is an amalgamation of the elements of K.

Conversely, let I be an amalgamation of the elements of K. Since $M(K)\supset K$, $I\in A(K)$. Let $\beta\in A(K)$; then there exists a subcollection of $M(K)$, M, such that $\beta<M<\beta$. For $m\in M$ there exists $k\in K$ such that $m<k$; by D 9.2 $M<K$; then $\beta<M<K<I$. Thus I is the unit of $A(K)$. By Theorem 18.2 and D 18.2 $A(K)+\overline{O}$ is a complete multiplicative system; since it has a unit, it is a complete lattice (cf. MacNeille, pp. 430–431).

Theorem 18.5. *If K is a collection of regular amalgamation points, and each element of K is a portion of some collection of regular composition points, then the additive and the multiplicative systems generated by K consist of regular points.*

Proof. If $k\in K$ and $\beta\in\alpha(k)$ it follows from the hypothesis and Theorem 9.2 that β is a portion of some regular composition point X. By Theorems 9.2, 14.2, 14.11, and 14.8 the greatest lower bound of k and X is a regular composition point $k(\beta)$, and $\beta<k(\beta)$. It can be shown with the help of Theorem 9.1 that $k<\alpha(k)<L<k$, where L is the aggregate of all $k(\beta)$'s. Thus, k is an amalgamation of a collection of regular composition points. The conclusion follows from D 18.3 and Theorems 14.11 and 14.10.

19. Systems of upper semi-continuous, perfectly compact decompositions of regular amalgamation points. Let us consider the aggregate $D(R)$ of all decompositions of the perfectly compact Hausdorff space R into upper semi-continuous collections of closed point sets. If X and Y are elements of $D(R)$ and each element of X is a subset of some element of Y, let us say that $X < Y$. This relation orders $D(R)$. The following analogue to Theorem 19.1 holds: If Y_1 and Y_2 are elements of $D(R)$ and X is the set of all products $y_1 \cdot y_2$, where $y_1 \in Y_1$ and $y_2 \in Y_2$, then X is the greatest lower bound of Y_1 and Y_2 in $D(R)$. Then X is upper semi-continuous, and it corresponds to the $\omega_M(P)$ of Theorem 19.1, if M is the pair Y_1 and Y_2. Such analogies may be extended to the case of arbitrary subcollections M of $D(R)$. We shall now extend these ideas to the case of the amalgamation points.

D 19.1. If P is a regular amalgamation point, let $\delta(P)$ be the set of all δ's, where δ is an upper semi-continuous decomposition(32) of P into regular amalgamation points. If X and Y are elements of $\delta(P)$, let the relation $X < Y$ mean that each element of Y is decomposible into a set of points which is a subcollection of X; let $\delta(P)$ be ordered by this relation. Let $L(P)$ denote the sum of the elements of $\delta(P)$, and $\omega(P)$ be the set of atomic elements of $L(P)$.

Clearly this relation partially orders $\delta(P)$. The relation $X < Y$ is merely the requirement that X and Y satisfy the conditions of Theorem 15.4. Thus, the study of $\delta(P)$ and its sublattices systematizes our information about the mappings and inverse mappings we considered in §§15 and 16. We shall show that $\delta(P)$ is a complete lattice and that $\omega(P)$ is its zero. Thus, $\omega(P)$ may be mapped on any element of $\delta(P)$ by the methods of Theorem 15.4; the elements of $\omega(P)$ may be regarded as the *atomic points* from the point of view of these mappings. The zero of a sublattice of $\delta(P)$ has an analogous relation to the elements of the sublattice; and the elements of this zero may be regarded as the atomic points of the mapping theory which involves the elements of this sublattice.

Example. E 19.1. Let P be an amalgamation point in a semi-completely normal space S or a boundary point of a normal space. Then $L(P)$ consists of all amalgamation points which are portions of P, and $\omega(P)$ is the decomposition of P into atomic points (cf. Theorems 16.1, 16.5, and 14.13).

D 19.2. If P is the maximal amalgamation point, let $\delta(S) = \delta(P)$ and $\omega(S) = \omega(P)$.

The preceding example suggests questions which the author has not solved for the case of the completely regular spaces. (I) If A and B are regular amalgamation points and $A < B$, does $\omega(B) \supset \omega(A)$? (II) Does $\omega(S)$ consist of the atomic regular points? An affirmative for (II) implies one for (I). A negative for (I) would imply the possibility of the existence of "incommensurable points," A and B; or that of a point P and a point β such that $\beta \in \omega(P)$ and $\omega(\beta)$ consists of proper portions of β. Theorem 14.12 shows that if S is not semi-completely normal, some of its atomic points are too fine to be regular; the preceding discussion suggests that for more general spaces S some of the atomic regular points may be too fine to belong to $\omega(S)$. In any case the existence of $\omega(P)$ is of importance, since it is the finest decomposition of P for which the methods of §15 are applicable.

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(32) By Theorem 15.3 the condition of the upper semi-continuity of δ is equivalent to that of its perfect compactness (cf. D 9.3).

THEOREM 19.1. *Let P be a regular amalgamation point which is a portion of a collection of regular composition points; let M be a subcollection of $\delta(P)$; let $L_M(P)$ be the additive system which is generated by the sum of the elements of M, and $\omega_M(P)$ be the set of atomic elements of $L_M(P)$:* (1) *$\omega_M(P) \in \delta(P)$;* (2) *if $\beta \in \omega_M(P)$ and $N \in M$, β intersects exactly one element of N, say N_β; and β is the greatest lower bound of all the N_β's;* (3) *$L_M(P)$ is the sum of all the Y's such that $Y \in \delta(P)$ and $\omega_M(P) < Y$ in $\delta(P)$; the elements of M are such Y's;* (4) *$L_M(P)+\bar{O}$ is a complete lattice of regular amalgamation points, and its unit is P.*

The lattice of (4) is obtained by the methods of the preceding section. The set of its atomic elements, $\omega_M(P)$, is the zero of a certain sublattice of $\delta(P)$; this lattice is suggested in condition (3) (cf. Theorem 19.3). This condition is used to show that the systems in the following two theorems are complete multiplicative systems.

Proof. Let β be an atomic portion of P; it intersects exactly one element, N_β, of each element N of M. By Theorem 9.2, $\beta < P_\beta$, the greatest lower bound of all the N_β's. By D 18.3 $P_\beta \in L_M(P)$; by Theorem 18.5 P_β is regular. By Theorems 18.1 and 18.2 there exists an atomic element of $L_M(P)$, say A, which is a portion of P_β; and there exists a subcollection of the sum of the elements of M, say A^*, such that A is the greatest lower bound of A^*. Since $A < P_\beta$, A is a lower bound of the elements of $[N_\beta]$. If $N \in M$, no two elements of N intersect, since N is a decomposition of P into amalgamation points. It follows that $[N_\beta] \supset A^*$; by Theorem 9.2 $P_\beta < A$; since $A < P_\beta$, $A = P_\beta$; thus we have established (2). Since $\beta < P_\beta \in \omega_M(P)$, it follows with the help of Theorems 9.1 and 9.2 that $P < \alpha(P) < \omega_M(P) < P$; by Theorem 18.1 no two elements of $\omega_M(P)$ intersect; thus $\omega_M(P)$ is a decomposition of P. It remains to show that $\omega_M(P)$ is upper semi-continuous.

First we shall establish this upper semi-continuity for the case that the number λ of elements of M is a positive integer. Let $n = \lambda$ be a positive integer such that the conclusion holds for each collection of n elements of $\delta(P)$; clearly, $n = 1$ is such an integer. Let M and H be two subcollections of $\delta(P)$ which have, respectively, $n+1$ and n elements, and are such that $M \supset H$. Let $J = \omega_H(P)$, K be the element of $M - H$, and L be the set of atomic elements of the additive system generated by $J+K$. It follows from part (2) and Theorem 9.2 that $L=\omega_M(P)$. Let $\beta\in L$ and let D be an S-neighborhood of β. There exist elements of J and K, say j and k, such that β is the greatest lower bound of j and k. By an argument used in the proof of Theorem 14.11 there exist in S mutually exclusive open sets D_1 and D_2 such that $D+D_1$ and $D+D_2$ are S-neighborhoods, respectively, of j and of k. Since J is upper semi-continuous, there exists an S-neighborhood of J, R_1, such that if R_1 is an S-neighborhood of a portion of an element of j, then $D+D_1$ is an S-neighborhood of that element. Similarly define R_2 for k and $D+D_2$; let $R=R_1\cdot R_2$. Let R be an S-neighborhood of a portion of an element of L, β_1, which is the greatest lower bound of j_1 and k_1. Then $D+D_1$ and $D+D_2$ are S-neighborhoods of j_1 and of k_1. Since $D_1\cdot D_2$ is vacuous, it follows from Theorems 9.2 and 11.6 that D is an S-neighborhood of $\alpha(\beta_1)=\alpha(j_1)\cdot\alpha(k_1)$ and of β_1. Thus L is upper semi-continuous, and our conclusion holds for any finite case.

Let M be infinite. Again let $\beta\in\omega_M(P)$ and D be an S-neighborhood of β. By (2) β is the greatest lower bound of the elements of $[N_\beta]$; by Theorem 9.2 $\alpha(\beta)$ is the product of the sets $\alpha(N_\beta)$. By methods used in the proof of Theorem 14.11 there exist a finite collection of the N_β's, $H=(N_{1\beta}, N_{2\beta}, \cdots, N_{k\beta})$ such that D is an S-neighborhood of the product $X=\alpha(N_{1\beta})\cdot\alpha(N_{2\beta})\cdots\alpha(N_{k\beta})$. If β_k is the greatest lower bound of the elements of H, by Theorems 9.2 and 11.6 D is an S-neighborhood of β_k. Let $N_{i\beta}\in N_i\in M$, $G=(N_1, N_2, \cdots, N_k)$, and $L=\omega_G(P)$. By the preceding paragraph $L\in\delta(K)$ and $\beta_k\in L$. Since L is upper semi-continuous, there exists an S-neighborhood of β_k, R, such that if R is an S-neighborhood of a portion of an element of L then D is an S-neighborhood of that element. By Theorem 9.2 $\beta<\beta_k$; by Theorem 11.6 R is an S-neighborhood of β. By condition (2) each element of $\omega_M(P)$ is a portion of an element of L; it follows that $\omega_M(P)$ is upper semi-continuous. We have established (1).

Let Z be the almost-multiplicative system which is generated by the sum of the elements of M. By Theorem 18.2 $\omega_M(P)$ is the set of atomic elements of Z. Let $Y\in\delta(P)$ and $\omega_M(P)<Y$ in $\delta(P)$; by D 19.1 the latter means that each element of Y, y, is an amalgamation of a subcollection of $\omega_M(P)$. Since $Z\supset\omega_M(P)$, $L_M(P)\supset Y$ (cf. D 18.3). Thus, if Σ is the sum of all Y's that satisfy the condition of (3), then $L_M(P)\supset\Sigma$. Conversely, let $A\in L_M(P)$ and let H be the set of all elements of $\omega_M(P)$ that are portions of A; then A is an amalgamation of the elements of H. Let L be the set which consists of H and the elements of $\omega_M(P)-H$; let Y consist of A and the elements of $\omega_M(P)-H$. Since $L_M(P)$ is an almost-multiplicative system and its atomic elements are the elements of $\omega_M(P)$, no element of $\omega_M(P)-H$ intersects A (cf. Theorems 18.1 and 18.2). By Theorem 14.7 A and H can be separated from each element of $\omega_M(P)-H$ by S-neighborhoods, and none of these elements is a limit point of H. Thus, H is closed relative to $\omega_M(P)$. By Theorem 15.3 $\omega_M(P)$ is perfectly compact in itself. Since L is an upper semi-continuous decomposition of $\omega_M(P)$ into closed point sets, it follows from Theorem 15.5 that Y is upper semi-con-

tinuous relative to S. Then $Y\in\delta(P)$ and $\omega_M(P)<Y$ in $\delta(P)$. Thus, $\Sigma\supset L_M(P)$; since $L_M(P)\supset\Sigma$, $L_M(P)=\Sigma$.

The conclusion of (4) follows from Theorems 18.4 and 18.5.

Theorem 19.2. *Let P be the point of Theorem 19.1. If $M=\delta(P)$, then $L(P)=L_M(P)$ and $\omega(P)=\omega_M(P)$. Also, $\delta(P)$ is a complete lattice with unit P and zero $\omega(P)$.*

Proof. If $M=\delta(P)$, by Theorem 19.1 and D 19.1 $L_M(P)\supset L(P)$, and conversely. Thus, $L_M(P)=L(P)$ and $\omega_M(P)=\omega(P)$. By Theorem 19.1, (3), if $Y\in\delta(P)$, then $\omega(P)<Y$ in $\delta(P)$. Thus, $\omega(P)$ is the zero of $\delta(P)$; clearly P is the unit of $\delta(P)$.

By Theorem 19.1 if $\delta(P)\supset M$ and $Y\in M$, then $\omega_M(P)<Y$ in $\delta(P)$. Let X be a lower bound of M in $\delta(P)$. If $x\in X$ and $N\in M$, there exists one element of N, N_x, such that x is a portion of N_x (cf. D 19.1). By (2) of Theorem 19.1 there exists an element of $\omega_M(P)$, say β_x, which is the greatest lower bound of the N_x's. Thus, x is a portion of β_x. From this relation and the fact that each of X and $\omega_M(P)$ is a decomposition of P, each element of $\omega_M(P)$ can be decomposed into a subcollection of X. Thus, $X<\omega_M(P)$ and $\omega_M(P)$ is the greatest lower bound of M in $\delta(P)$. Since $\delta(P)$ has a unit, it is a complete lattice (cf. MacNeille, pp. 430, 431, and Birkhoff, p. 17).

Theorem 19.3. *If $\delta(P)\supset M$ and P satisfies the condition of Theorem 19.1, let $\omega_1(M, P)=\omega(P)$, $\omega_2(M, P)=\omega_M(P)$, $\omega_3(M, P)$ be the least upper bound of M in $\delta(P)$, and $\omega_4(M, P)=P$; if $1\leqq i\leqq j\leqq 4$, let $\delta_{ij}(M, P)$ be the set of all elements of $\delta(P)$, X, such that $\omega_i(M, P)<X<\omega_j(M, P)$ in $\delta(P)$. Then $\delta_{ij}(M, P)$ is a complete sublattice of $\delta(P)$ and its zero and unit are, respectively, $\omega_i(M, P)$ and $\omega_j(M, P)$; $\omega_M(P)$ is the greatest lower bound of M in $\delta(P)$.*

Thus, any $X<\omega_2(M, P)$ may serve as a common inverse mapping space for the elements of M. However, $X=\omega_2(M, P)$ may be regarded as the "most economical" of these inverse spaces. For, if $X\in\delta_{12}(M, P)$ and Y_1 and Y_2 are elements of M, we can first decompose the elements of Y_1 into points of X and then reamalgamate these into points of Y_2 by the methods of Theorem 15.4. Since the elements of $X=\omega_2(M, P)$ are "coarser" than those of any other element of $\delta_{12}(M, P)$, the initial decomposition need not be extended so far if this X is used for an inverse space. Similarly, $X=\omega(P)$ is the "least economical" of these inverse spaces, but it serves for all sets in $\delta(P)$. Analogous interpretations hold for the common mapping spaces of the elements of M, the elements of $\delta_{34}(M, P)$. Theorem 19.3 may be modified by replacing $\delta(P)$ by any of its complete sublattices.

20. **Applications to completely regular spaces.** If we order all the perfectly compact immediate Hausdorff extensions of S by the ordering of D 19.1, we obtain a quasi-partially ordered system (cf. Theorems 13.7 and 16.1). By

identifying(33) equivalent elements of this system, we obtain a complete multiplicative system which is isomorphic to the subsystem of $\delta(S)$, $H(S)$, which we consider in D 20.1 and Theorems 20.2 to 20.4. The zero of $H(S)$, $\lambda(S)$, is equivalent to the space considered by Stone in his Theorem 88, p. 476. In §17 we considered this space for the case that S is normal (cf. also, Čech, loc. cit.)

The important theorems 14.11, 18.5, and 19.1 to 19.3 involve hypotheses in which certain points satisfy the condition of being the amalgamation of a collection of regular composition points(34). This condition and the results of the preceding section enter into the characterizations of completely regular spaces which are given in Theorem 20.1. The fact that the atomic mapping points in such a space are necessarily regular composition points gives an analogy to the results of Theorem 14.13.

D 20.1. Let $H(S)$ be the subsystem of $\delta(S)$ which consists of all Y's such that each point of Y is either a boundary point or is equivalent(35) to a point of S. Let $\lambda(S)$ be the zero of $H(S)$ (cf. D 19.2).

THEOREM 20.1. *Any of the following is a necessary and sufficient condition that S be completely regular*: (1) *At least one element of* $\delta(S)$ *consists of composition points*; (2) *the elements of* $\omega(S)$ *are composition points*; (3) *if* Q *is a regular amalgamation point, the elements of* $\omega(Q)$ *are regular composition points.*

Any regular amalgamation point in a completely regular space is the amalgamation of some collection of regular composition points(36).

Proof. By Theorems 13.7 and 16.1 the condition in (1) is necessary. Let Y be an element of $\delta(S)$ which consists of regular composition points, Q be a regular amalgamation point, and K be the aggregate of all points Q_y, where Q_y is the greatest lower bound of Q and the element y of Y. By Theorem 14.11 if y intersects Q, then Q_y is a regular amalgamation point. Since Y is a decomposition of the maximal amalgamation point P, and $Q<P$, it may be shown that K is a decomposition of Q. We may regard Q as an upper semi-continuous collection of one element; since Y is upper semi-continuous, we may show by an argument like that used for the finite case in the proof of Theorem 19.1 that K is upper semi-continuous relative to S. By D 19.1 $K\in\delta(Q)$. By Theorem 19.1 if $\beta\in\omega(Q)$, there exist elements of K and of Y, respectively, k_β and y_β, such that $\beta<k_\beta<y_\beta$. Since β is a regular amalgamation point and y_β is a regular composition point, it follows from Theorem 14.8 that β is a composition point. Thus, the conditions in (2) and (3) are necessary. Since $\omega(Q)\in\delta(Q)$, by D 19.1, D 14.2, and D 9.3 Q is an amalgamation of $\omega(Q)$.

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(33) Cf. Birkhoff, p. 7, Theorem 1.2.

(34) There exist examples which show that such an hypothesis is essential for the truth of these theorems.

(35) Cf. D 3.2 and Theorem 13.7.

(36) Cf. Theorem 14.13.

Clearly, each of (2) and (3) implies (1) (cf. Theorem 14.5). Let X be an element of $\delta(S)$ which consists of regular composition points, M be the set of those elements of X which are boundary points, and $Y=S+M$. By Theorem 5.2 no boundary point intersects a decomposition point. By D 19.1, D 19.2, and D 9.2 each boundary point intersects M. By the argument used at the beginning of the proof of Theorem 16.1 it follows that the points of S are regular. Since no two elements of Y intersect, it follows from Theorems 14.6, 14.7, and 12.1 that Y is a Hausdorff space. Let the mapping $y=\alpha(x)$ be defined as in Theorems 15.4 and 15.6. Let $\beta \in Y$, and D be an S-neighborhood of β. Since β is regular, there exists an S-neighborhood of β, R, such that $D \supset \overline{R}_S$. Let x be an element of X such that R is an S-neighborhood of x. By Theorem 12.4 $\overline{R}_S \supset a(x)$. By D 12.2 and the definition of the mapping, $y=\alpha(x)$, either $x=\alpha(x)$ or $a(x)=\alpha(x)$; in either case D is an S-neighborhood of $\alpha(x)$. Thus, our mapping is continuous from X to Y (cf. Alexandroff and Hopf, p. 53, Theorem IV). By Alexandroff and Hopf, Theorem VIII, p. 98, Y is perfectly compact. By Theorems 15.3 and 16.1 the condition in (1) is sufficient.

THEOREM 20.2. *Let S be completely regular*: (1) $H(S)$ *is a complete multiplicative subsystem of* $\delta(S)$; (2) $\lambda(S)$ *consists of the composition points which are equivalent to the points of* S, *and of the boundary points which are elements of* $\omega(S)$; (3) *if* $X=\lambda(S)$ *and* $Y \in H(S)$, *then* X *may be mapped on* Y *by the mapping* $y=\alpha(x)$ *of Theorem* 15.4.

Concerning (3) cf. Stone, loc. cit., Theorem 88.

Proof. Let H' denote the set of composition points which are equivalent to points of H (cf. Theorem 13.7 and D 3.2). Let M be a subcollection of $H(S)$. By Theorem 19.3 $\omega_M(S)$ is the greatest lower bound of M in $\delta(S)$. Since S' is a subset of each element of M, and the remaining points of each element of M are boundary points, by Condition (2) of Theorem 19.1 $\omega_M(S) \supset S'$, and $\omega_M(S)-S'$ consists of boundary points (cf. Theorem 14.8). Thus, $\omega_M(S) \in H(S)$; (1) follows by D 3.7. If $M=H(S)$, by Theorem 19.3 and D 20.1 $\lambda(S)=\omega_M(S)$. Let K be the set of all boundary points which belong to $\omega(S)$, and let $Y=S'+K$. By an argument similar to that used in the proof of the sufficiency of the condition of Theorem 20.1 it follows that $Y \in \delta(S)$; hence, $Y \in H(S)$. Thus, $\lambda(S) < Y$ in $H(S)$. Since $\omega(S) < \lambda(S)$ in $\delta(S)$, it follows that $\lambda(S)-S'=K$, and that $\lambda(S)=Y$.

D 20.2. Let the symbol H' denote the set of all composition points which are equivalent to elements of H (cf. D 3.2 and Theorem 13.7). Let B be the set of all boundary points and C be the set of all real points which are limit points of B. Let $Q(S)$ be the amalgamation of the elements of $B+C$. Let $I(S)=(S'-C')+Q(S)$.

Recall that $H(S)$ contains the topological images of all immediate, perfectly compact Hausdorff extensions of S. Theorem 20.2 shows that $\lambda(S)$ is a universal inverse mapping space for such extensions, and is one of them. The two following theorems show that $I(S)$ is the least common mapping space of the elements of $H(S)$, and that it does not necessarily belong to $H(S)$. The following theorem gives the structure of $I(S)$. In particular, if S is not locally perfectly compact at any of its points, then $Q(S)$ is $I(S)$ and is the maximal amalgamation point. Then $Q(S)$ is the unit of $\delta(S)$.

THEOREM 20.3. *Let S be completely regular.* (1) $I(S)$ *is the least upper bound of* $H(S)$ *in* $\delta(S)$; (2) $I(S)-Q(S)$ *consists of those composition points which are equivalent to the points of S at which S is locally perfectly compact.*

Proof. Let M be the set of all boundary points, N be the set of real points which are limit points of M, and $K=S-N$. If $X\in K$, there exists an S-neighborhood of X, R, which is not an S-neighborhood of any element of $M+N$ (cf. Theorems 12.2 and 13.1). Let D be an S-neighborhood of X such that $R\supset\overline{D}$. If $\overline{D}$ were not perfectly compact, there would exist a boundary point β such that β is an end of $\overline{D}$ (Theorem 10.3). Then, since $R\supset\overline{D}$, it may be shown with the help of Theorem 10.1 that R is an S-neighborhood of β; since $\beta\in M$, we are involved in a contradiction. It follows that if the composition point δ is a limit point of $M+N$, then $X\neq a(\delta)$ (cf. D 12.2). Thus, either $a(\delta)\in N$, or $a(\delta)$ is vacuous, and δ is a boundary point; in either case δ intersects an element of $M+N$. By Theorems 14.1 and 13.7 $Q(S)$ is an amalgamation point. Clearly, no two elements of $I(S)$ intersect, each ideal point intersects an element of $I(S)$, and $I(S)$ is upper semi-continuous relative to S (cf. D 15.1 and D 19.2). Thus, $I(S)\in\delta(S)$.

Let $Y\in H(S)$ and Z be an upper bound of $H(S)$ in $\delta(S)$. Clearly, $Y<I(S)$ in $\delta(S)$. If A and B are boundary points which belong to elements of $H(S)$, they can be decomposed into subcollections of $\lambda(S)$, H_A and H_B; and they can be amalgamated into a point C such that $C<H_A+H_B<C$. Let $L=C+\lambda(S)-(H_A+H_B)$. Then C intersects no element of $L-C$; since $\lambda(S)$ contains $L-C$ and is upper semi-continuous, it follows that L is upper semi-continuous; also, L is a decomposition of the maximal amalgamation point. Thus, L is an element of $\delta(S)$ and of $H(S)$, and $L<Z$. Let $\delta\in Z$ such that C is a portion of δ; since A and B are arbitrary, it follows that δ is independent of A, B, and C; thus, by D 9.2 M is a portion of δ. Since the elements of N are limit points of M, they intersect δ (cf. Theorem 14.6). Since $S'\supset N'$, each element of S' belongs to each element of $H(S)$, and no two elements of Z intersect, each element of N is a portion of some element of Z; *that is, of* δ. By D 9.2 $Q(S)$ is a portion of δ. Since the elements of $I(S)-Q(S)$ belong to S', it follows that $I(S)<Z$ in $\delta(S)$. We have established the conclusion.

THEOREM 20.4. (1) *In order that* $H(S)$ *be a complete sublattice of* $\delta(S)$, *it is necessary and sufficient that S be locally perfectly compact.* (2) *If this condition is satisfied,* $\lambda(S)$ *and* $I(S)$ *are the zero and the unit of* $H(S)$; *and* $Q(S)$ *is the amalgamation of all the boundary points.*

Here $I(S)$ is the topological image of the space we considered in Theorem 16.2. Cf. Alexandroff and Hopf, p. 93.

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ON EXTENDING A CONTINUOUS (1-1) CORRESPONDENCE OF TWO PLANE CONTINUOUS CURVES TO A CORRESPONDENCE OF THEIR PLANES

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Reprinted from the
TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY
Vol. XXVIII, No. 2, pp 252-265
April, 1926

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Various authors have considered the following problem: given two sets of points, M and M', lying in planes S and S' respectively, and a continuous (1-1) correspondence† T, such that $T(M)=M'$, under what conditions can the correspondence be extended to the planes? That is, under what conditions does there exist a continuous (1-1) correspondence U, such that $U(S)=S'$, and such that for points of M, U is identical with T?

A. Schoenflies‡ has shown that in case M is a simple closed curve the correspondence can be extended to the planes, without any conditions being imposed.

R. L. Moore§ has shown that if M and M' are subsets of arcs, the correspondence can always be extended to the planes. If we consider only the case where M is a connected set, Moore's theorem applies only to the case where M is an arc.

Moore and Schoenflies, then, have proved that if M is an arc or a simple closed curve, the correspondence can be extended to the planes, without any conditions being imposed on the correspondence. In this paper, we show that if M is any plane continuous curve,|| the correspondence can

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* Presented to the Society, May 3, 1924, and February 28, 1925; received by the editors in March, 1925.

† A correspondence T which sends M into $T(M)$ is said to be *continuous*, if in case the point P of M is a limit point of N, a subset of M, then $T(P)$ is a limit point of $T(N)$. See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), p. 289. We shall refer to this paper hereafter as "Report."

‡ A. Schoenflies, *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), p. 324. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1920), p. 529. A *simple closed curve* is a set which is in continuous (1-1) correspondence with a circle.

§ R. L. Moore, *Conditions under which one of two given closed linear point sets may be thrown into the other one by a continuous transformation of a plane into itself* American Journal of Mathematics, vol. 48 (1926), p. 67. An *arc* is a set which is in continuous (1-1) correspondence with an interval of a straight line.

|| For the various definitions of a continuous curve see Report, pp. 289-295.

be extended to the planes, provided that we impose the condition that sides of arcs be preserved under the correspondence. Our theorem includes Schoenflies's theorem as a special case, since our condition is evidently satisfied in this special case.

The following example shows the necessity for imposing some condition on the correspondence T. The continuous curve M in the XY plane consists of the straight line intervals from (0, 0) to (3, 0), from (1, 0) to (1, 1) and from (2, 0) to (2, 1). The continuous curve M' in the $X'Y'$ plane is given by subjecting the points of M to the following transformation: if $x \neq 2$, $x = x'$ and $y = y'$; if $x = 2$, $x = x'$ and $y = -y'$. Here T is a continuous (1-1) correspondence and $T(M) = M'$, but the correspondence evidently cannot be extended to the entire planes.

Definition. If M and M' are continuous curves lying in planes S and S' respectively, and T is a continuous (1-1) correspondence such that $T(M) = M'$, we say that *sides are preserved under* T, if, given any arc AB of M, and any simple closed curve J in S containing AB as a subset, then there exists a simple closed curve J' in S' containing $T(AB) = A'B'$ as a subset, and such that if N designates the points of M interior to J, then the interior of J' contains $T(N) = N'$; and also, if given any simple closed curve J_1' in S' containing $A'B'$ as a subset, then there exists a simple closed curve J_1 in S containing AB as a subset, and such that if N_1' designates the points of M' interior to J_1', then the interior of J_1 contains $T^{-1}(N_1') = N_1$.

In the following we shall frequently use this notation: if X is any subset of M, we shall denote $T(X)$ by X'; if Y' is any subset of M', we shall denote $T^{-1}(Y')$ by Y.

Theorem I. *If M and M' are continuous curves containing no simple closed curve** *and lying in planes S and S' respectively, and if there exists a continuous* (1-1) *correspondence T, such that $T(M) = M'$, and such that sides are preserved under T, then there exists a continuous* (1-1) *correspondence U, such that $U(S) = S'$, and such that if, for any point P of M, $T(P) = P'$, then $U(P) = P'$.*

Before proceeding with the proof of Theorem I, we shall discuss the definition of "sides preserved under T" for the case where M is a continuous curve containing no simple closed curve. In this discussion we have need of the following lemma.

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* For a discussion of this type of continuous curve, see S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), p. 119; R. L. Wilder, abstracts in the Bulletin of the American Mathematical Society, vol. 29 (1923), p. 118, and *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 340; and R. L. Moore, *Concerning the cut-points of continuous curves*, Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 101.

LEMMA A. *If I_1 and I_2 are two simply connected domains, whose boundaries are B_1 and B_2, and whose outer boundaries*[*] *C_1 and C_2 have in common an arc UV such that no point of $\underline{UV}$*[†] *is a limit point of $B_1+B_2-\underline{UV}$, then $\underline{UV}$ is in the boundary of two different domains complementary to B_1+B_2, such that either* (1) *one domain is a subset of I_1, the other of I_2, or* (2) *one domain is a subset of both I_1 and I_2, and the other has no points in common with either I_1 or I_2.*

Proof. About each point X of $\underline{UV}$, let us construct a circle C_X whose exterior contains $B_1+B_2-\underline{UV}$. Then corresponding to each point X, we can construct a simple closed curve J_X, formed of an arc X_1XX_3 of $\underline{UV}$ and an arc $\underline{X_1X_2X_3}$ in I_1, and whose interior is in I_1 and the interior of C_X.[‡] The sum of the interiors of the simple closed curves J_X is a domain D, because if the boundaries of any pair have an arc of $\underline{UV}$ in common, their interiors have a point in common. The domain D contains no points of B_1+B_2 by construction. If we add to D all points which can be joined to a point of D by an arc having no points in common with B_1+B_2, we obtain a domain D_1 complementary to B_1+B_2, and such that $\underline{UV}$ forms part of the boundary of D_1. Evidently D_1 is a subset of I_1.

By a construction similar to the above, but taking in this case the arc $\underline{X_1X_2X_3}$ exterior to I_1, we obtain a domain D_2 complementary to B_1+B_2, whose boundary contains $\underline{UV}$, and which has no points in common with I_1.

Since $\underline{UV}$ is part of the boundary of I_2, either D_1 or D_2 contains a point of I_2. Since a point of I_2 cannot be joined to a point exterior to I_2 by an arc having no points in common with B_2, it is evident that either D_1 or D_2 is a subset of I_2, and the other has no points in common with I_2. We have accordingly, the two possibilities mentioned in Lemma A.

We shall now consider some consequences of the definition of "sides are preserved under T" for a continuous curve M containing no simple closed curve. Let AB be a maximal arc[§] in M. Let J be a simple closed curve in the plane S, containing AB as a subset and containing no other points of M.

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[*] R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 254.

[†] If UV is an arc, $\underline{UV}$ denotes $UV-U-V$.

[‡] R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916) p. 131. See especially Theorem 28.

[§] A *maximal arc* in a set M is an arc which is not a proper subset of any other arc in M. See S. Mazurkiewicz, loc. cit., Lemma 13, p. 129.

Let N be the points of M interior to J. Since sides are preserved under T, there exists in S' a simple closed curve J', containing $A'B'$ as a subset, and enclosing N'. We shall now show that in case J' contains or encloses any points of $M'-(A'B'+N')$, J' can be replaced by a simple closed curve J'' which contains $A'B'$ as a subset, and which encloses N', but which neither contains nor encloses any other points of M'.

We shall first show that A' and B' are on the boundary of the same domain of $(I'+M')-M'$, where I' designates the interior of J'. For, if not, there is an arc $C'D'$ in M', such that (a) C' is on $J'-A'B'$; (b) D' is on $\underline{A'B'}$; (c) $\underline{C'D'}$ is in I'. Since N' is entirely in I', C' is not a point of N'. Since $N+AB$ is closed, $N'+A'B'$ is closed, and therefore no point of $\underline{C'D'}$ can be in the set N'.

Now let J' be the J_1' of the definition. The curve J' encloses N' and $\underline{C'D'}$. The corresponding simple closed curve J_1 in S encloses N and $\underline{CD}$, and contains the arc AB as a subset. The simple closed curves J and J_1 in S have AB in common, and therefore satisfy the conditions of Lemma A. Since their interiors have N in common, and since limit points of N are on $\underline{AB}$, it follows that one of the domains (complementary to $J+J_1$), whose boundary contains $\underline{AB}$, consists entirely of points common to the interiors of J and J_1. Since D is a point of $\underline{AB}$, and since $\underline{CD}$ is interior to J_1, it follows that at least part of $\underline{CD}$ is interior to J and therefore in the set N. In that case, the corresponding part of $\underline{C'D'}$ lies in N', contrary to a previous statement. Therefore A' and B' are on the boundary of the same connected domain of $(I'+M')-M'$.

The points A' and B' can therefore be connected by an arc[*] in $I'-M'$, which forms with the arc $A'B'$ of M' a simple closed curve J''. The arc $J''-\underline{A'B'}$ separates I' into two parts, and the part enclosed by J'' contains all of N'. The supposition that J'' encloses other points of M' in addition to N' leads to a contradiction similar to that obtained above.

Therefore in case M contains no simple closed curve, and AB is a maximal arc of M, and J contains AB but no other points of M, then we can add to our definition of "sides are preserved under T," that J' contains $A'B'$ but no other points of M', and that the interior of J' contains N' and no other points of M'; and similarly for J_1' and J_1.

Lemma A and the previous discussion show also that if any two simple closed curves have a maximal arc AB of M in common and contain no other points of M, then either their interiors contain the same subset of M, or their interiors have no points of M in common, in which case, since $M-A-B$ is connected, they contain all of M, save the arc AB.

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[*] Report, pp. 290–291.

DEFINITION. If M is a continuous curve containing no simple closed curve, and N is a closed and connected subset of M, we shall call a maximal connected subset of $M-N$ a *tree with respect to* N, or a *tree in* $M-N$. A tree has one and only one limit point in N, which point we shall call the *foot* of the tree. A tree plus its foot forms a closed set.

R. L. Wilder* has proved that the number of trees is countable, and that given any positive number ϵ, there are at most a finite number of trees of diameter greater than ϵ.

Proof of Theorem I. Since M and M' are bounded we can construct in the plane S a circle C containing M in its interior I, and in the plane S' a circle C' containing M' in its interior I'. If AB is a maximal arc of M, we can join A to any point D of C by an arc in $(I-M)+A+D$, and we can join B to any other point E of C by an arc in $(I-M)+B+E-AD$. The arc $A'B'$ in M' is also a maximal arc, and if we select any arbitrary points D' and E' of C', we can join A' to D' and B' to E' by arcs in $(I'-M')+A'+D'$ and $(I'-M')+B'+E'-A'D'$, respectively.

If X and Y are points of C separating D and E, the arcs EXD (of C), DA, AB (of M), and BE form a simple closed curve J, and the arcs EYD (of C), DA, AB and BE form a simple closed curve J_1. The interiors of J and J_1 have no points in common, and the sum of their interiors contains all points of M save AB.

If X' and Y' are any two points of C' separating D' and E', there exist likewise in S' two simple closed curves $J'=E'X'D'A'B'E'$ and $J_1'=E'Y'D'A'B'E'$, whose interiors have no points in common, and the sum of whose interiors contains all points of M' save $A'B'$.

In our previous discussion of "sides are preserved under T," we have shown that under the above conditions, one of the simple closed curves J' or J_1' (suppose the former) will enclose all the points of M' which correspond under T to the points of M which J encloses, and the other, J_1', will enclose the points of M' which correspond to the points of M which J_1 encloses.

Let us select an arbitrary positive number ϵ. Suppose either $M-AB$ or $M'-A'B'$ contains a tree of diameter greater than $\frac{1}{9}\epsilon$. Let T be a tree in $M-AB$ which is interior to J, and let T' be the corresponding tree in $M'-A'B'$ which is interior to J', these trees being such that the diameter of either T or T' is greater than $\frac{1}{9}\epsilon$. Let the foot of T be F, and let FG be a maximal arc of $(T+F)$ such that the diameter of either FG or $F'G'$ is greater than $\frac{1}{18}\epsilon$. If H is any point of $\underparen{DXE}$, H can be joined to G by an arc interior to J, save for H, and having only G in common with M. If H' is any point of $\underparen{D'X'E'}$ a similar arc $H'G'$ can be drawn. The arc FGH separates the interior of J into two domains, and the arc $F'G'H'$ similarly separates the interior of J'. It is evident that the points of M interior to the simple closed curve $FGHDAF$ have their corresponding points in M' interior to the simple closed curve $F'G'H'D'A'F'$. Similarly for points interior to $FGHEBF$.

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* R. L. Wilder, loc. cit., first paper, Theorem II.

If $M-(AB+FG)$ or $M'-(A'B'+F'G')$ contains a tree of diameter greater than $\frac{1}{9}\epsilon$, the simple closed curves enclosing it and the corresponding tree in the other set can both be separated by arcs in the manner indicated above. After a finite number of steps $M-(AB+FG+\cdots)$ and $M'-(A'B'+F'G'+\cdots)$ will contain no trees of diameter greater than $\frac{1}{9}\epsilon$, otherwise a theorem* due to R. L. Wilder is contradicted. At this stage, this state of affairs exists: the interior of the circle C is divided into a finite number of domains plus boundary points of these domains, where the domains are such that they are bounded by simple closed curves, each of which consists of a single maximal arc of M, and a single arc having no points in common with M save its end points; the interior of C' is divided into the same number of domains plus boundary points, where the domains are bounded by simple closed curves, each of which consists of a single maximal arc of M' and a single arc having no points in common with M' save its end points; if N represents the set of points of M contained and enclosed by one of the simple closed curves in the plane S, the set $T(N)=N'$ will be contained and enclosed by one of the simple closed curves in the plane S', and this simple closed curve will contain or enclose no points of M' which are not in N'; no tree in N (or N') with respect to the maximal arc of M (or M') belonging to the simple closed curve which contains and encloses N (or N'), is of diameter greater than $\frac{1}{9}\epsilon$.

Let K be any one of the simple closed curves in S; PQ the maximal arc of M on K; N the subset of M contained and enclosed by K. Let K' denote the corresponding simple closed curve in S'. We can select a finite set of points $P_1, P_2, \cdots, P_n$, and $P'_1, P'_2, \cdots, P'_n$, such that (a) $P_1=P$ and $P'_1=P'$; (b) $P_n=Q$ and $P'_n=Q'$; (c) P_i precedes P_{i+1} on PQ, and P'_i precedes P'_{i+1} on $P'Q'$, for $i=1, 2, \cdots, n-1$; (d) the diameter of each of the arcs P_iP_{i+1} and $P'_iP'_{i+1}$ is less than $\frac{1}{9}\epsilon$; (e) no tree in $N-PQ$ or $N'-P'Q'$ has its foot at any of the points P_i or P'_i; (f) $T(P_i)=P'_i$. Let the set K_i

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* R. L. Wilder, loc. cit., second paper, Theorem II.

$(i=1, 2, \cdots, n-1)$ be P_iP_{i+1} plus all trees in $N-PQ$ with feet on P_iP_{i+1}; and let K'_i be $P'_iP'_{i+1}$ plus all trees in $N'-P'Q'$ with feet on $P'_iP'_{i+1}$. Evidently $T(K_i)=K'_i$, and the diameter of each of the sets $K_1, K_2, \cdots, K_{n-1}, K'_1, K'_2, \cdots, K'_{n-1}$ is less than $\frac{1}{3}\epsilon$.

The point P_1 can be joined to P_2 by an arc in the interior of K, having only its end points in common with M. We shall show that under our given conditions P_1 can be joined to P_2 by such an arc whose diameter is less than $\frac{2}{3}\epsilon$. Suppose an arc P_1RP_2 has been constructed whose diameter is greater than $\frac{2}{3}\epsilon$. This arc forms with P_1P_2 a simple closed curve H which encloses $K_1-P_1P_2$, but encloses no other points of M. There also exists a simple closed curve L, enclosing K_1, and such that every point of L plus its interior is at a distance less than $\frac{1}{12}\epsilon$ from K_1,* and therefore such that the diameter of L is less than $\frac{1}{2}\epsilon$. Since the diameter of the arc P_1RP_2 is greater than $\frac{2}{3}\epsilon$, it must contain points exterior to L. The simple closed curves H and L satisfy the conditions† under which there exists a simple closed curve Q which is a subset of $H+L$, contains the arc P_1P_2 of M, and every point of whose interior is interior to both H and L. The arc from P_1 to P_2 in Q which has only its end points in common with M, has no points exterior to L, and is therefore of diameter less than $\frac{2}{3}\epsilon$; it has no points exterior to H and except for its end points, has no points in common with K_1, and therefore has no points in common with M, and lies entirely in the interior of K. This is an arc satisfying the conditions stated.

The point P_1 can therefore be joined to P_2 by an arc $P_1W_1P_2$ in the interior of K which forms with the arc P_1P_2 of M a simple closed curve C_1, enclosing $K_1-P_1P_2$ but no other points of M, and containing P_1P_2 but no other points of M, and such that the diameter of C_1 is less than ϵ. In the same way an arc $P_2W_2P_3$ can be constructed in the interior of K (save for P_2 and P_3) and exterior to C_1 (save for P_2), which forms with P_2P_3 of M a simple closed curve C_2, enclosing and containing the set K_2 and no other points of M and such that the diameter of C_2 is less than ϵ. In this way we construct the simple closed curves $C_1, C_2, \cdots, C_{n-1}$, all of diameter less than ϵ. In the same way we construct in S' the simple closed curves $C'_1, C'_2, \cdots, C'_{n-1}$, all of diameter less than ϵ.

Note that if from the interior of K we remove the simple closed curves $C_1+C_2+\cdots+C_{n-1}$ and their interiors, there remains a domain whose boundary is the simple closed curve consisting of the arcs $K-\underline{P_1P_n}$, $P_1W_1P_2$, $P_2W_2P_3, \cdots, P_{n-1}W_{n-1}P_n$; similarly for K'.

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* R. L. Moore, *Concerning the separation of point-sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), p. 469.

† R. L. Moore, *On the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry*, American Journal of Mathematics, vol. 41 (1919), p. 299. See especially Theorem 26.

If the interior of any of the simple closed curves $C_1, \cdots, C_{n-1}$ contains points of M, we shall separate its interior as we have separated the interiors of J and J' into a finite set of domains free from points of M, plus a finite set of simple closed curves (such as C_1) of diameter less than $\frac{1}{2}\epsilon$, the same being true of M'. We shall then continue this process with such of these simple closed curves as enclose points of M.

We shall now define a continuous (1-1) correspondence U, such that $U(S)=S'$, and we shall show that if, for any point P of M, $T(P)=P'$, then $U(P)=P'$.

For points of the circles C and C', U is any continuous (1-1) correspondence between C and C' subject only to these conditions: (1) $U(D)=D'$; $U(E)=E'$; (2) if P is a point of $\underline{EXD}$, $U(P)$ is on $\underline{E'X'D'}$; (3) if P is a point of $\underline{EYD}$, $U(P)$ is on $\underline{E'Y'D'}$; (4) $U(H)=H'$; and similarly for points of C other than H that were joined by arcs to points on trees of M of diameter greater than $\frac{1}{9}\epsilon$, or such that the corresponding trees in M' were of diameter greater than $\frac{1}{9}\epsilon$.

Having defined U for points of the circles C and C', we define U for points exterior to these circles, as being any continuous (1-1) correspondence between S and S' subject only to the condition that for points of C and C' the correspondence be the one defined in the preceding paragraph. That such a correspondence exists, follows from Schoenflies's theorem.

For points interior to C, and on one of the simple closed curves K, consisting of an arc PQ of M, an arc QX interior to C save for X, an arc XY on C, and an arc YP interior to C save for Y, we define U as being any continuous (1-1) correspondence between K and K' subject only to the following conditions: (1) for points of PQ, U is identical with T; (2) for points of XY, U is identical with the correspondence previously defined for points of C.

For points interior to one of the simple closed curves K, and on the arc $P_1W_1P_2W_2P_3\cdots P_{n-1}W_{n-1}P_n$, we define U as being any continuous (1-1) correspondence between $P_1W_1\cdots P_n$ and $P_1'W_1'\cdots P_n'$ subject only to the following conditions: $U(P_1)=P_1'\cdots$, $U(P_n)=P_n'$; and if C_i encloses points of M, the correspondence between $\underline{P_iW_iP_{i+1}}$ and $\underline{P_i'W_i'P_{i+1}'}$ must be such that $U(H_1)=H_1'$, where H_1 is a point of $\underline{P_iW_iP_{i+1}}$ which was joined, by an arc interior to C_i, to a point of a tree, in $M-P_iP_{i+1}$, of diameter greater than $\frac{1}{18}\epsilon$.

For points interior to the simple closed curve formed by $K-\underline{P_1P_n}$, and $P_1W_1P_2, \cdots, P_{n-1}W_{n-1}P_n$, U is defined as being any continuous (1-1) correspondence between S and S' subject only to the condition that for points of the simple closed curves, the correspondence be the one defined in the preceding paragraphs.

If the interior of any of the simple closed curves $C_1, C_2, \cdots, C_n$, say C_1, is free from points of M, U is defined for points of the interior of C_1 as being any continuous (1-1) correspondence between S and S' subject only to the condition that for points of C_1 and C_1' the correspondence be the one already defined for points of C_1 and C_1'.

In case the interior of any of the simple closed curves $C_1, \cdots, C_n$ contains points of M, we have already indicated the method of separation of its interior into a finite number of domains, and we have indicated above how U is defined for every point of S, save for points interior to simple closed curves of diameter less than ϵ, enclosing points of M. We shall now show that by a continuation of the above process, U is defined for such points.

If P is any point of $S-M$, there is some value of k such that $\epsilon/2^k$ is less than the distance from P to a point of M. Therefore at the step where the diameters of the simple closed curves enclosing points of M are less than $\epsilon/2^k$, P will not lie in the interior of such a simple closed curve, and hence U will have been defined for the point P. Similarly for any point in $S'-M'$.

If P is a point of M, and if U has not already been defined for P, then P lies at each step in the interior of a simple closed curve, consisting of an arc of M and an arc in $S-M$. Let the sequence of simple closed curves be $D_1, D_2, \cdots$, where D_{i+1} lies in D_i plus its interior; let the diameter of D_i be less than $\epsilon/2^i$; and let the arc of M on D_i be X_iY_i. Evidently P is the only limit point of this sequence. The corresponding simple closed curves in S' are $D_1', D_2', \cdots$, and D_{i+1}' lies in the interior of D_i', the diameter of D_i' is less than $\epsilon/2^i$, and the arc of M on D_i' is $X_i'Y_i'$, where $X_i'=T(X_i)$, and $Y_i'=T(Y_i)$. The sequence $D_1', D_2', \cdots$ approaches a single point P' as a sequential limit point, and we define $U(P)$ as P'.

It remains to be shown that $T(P)$ is the point $U(P)=P'$. The points $X_1, X_2, \cdots$ of M approach P as a sequential limit. Since T is continuous, the point in M' approached by $X_1', X_2', \cdots$ is $T(P)$, and we have called this point $U(P)=P'$. Therefore U is identical with T for all points of M.

The correspondence U as defined above satisfies the conditions of Theorem I.

DEFINITION. If M and M' are continuous curves lying in planes S and S' respectively, and T is a continuous (1-1) correspondence such that $T(M)=M'$, we say that *interiors are preserved under* T if, given any simple closed curves J of M and J' of M', such that $T(J)=J'$, and if N is the set of points of M interior to J, and N' is the set of points of M' interior to J', then $T(N)=N'$.

Lemma B. *If sides are preserved under T, interiors are preserved under T.*

Proof. Given $T(M)=M'$ and sides are preserved under T. Suppose P, a point of M interior to J, is such that $T(P)=P'$ is exterior to J'. We shall show that this leads to a contradiction.

Let PQ be an arc in M, having only Q in common with J, and therefore interior to J, save for Q. Then $P'Q'$ will be exterior to J', save for Q'.

Let A and B be two points of J, $A\neq Q\neq B$, and let D be a point of J, such that D and Q separate A and B. In the exterior of J we shall construct an arc $\underset{\smile}{AXB}$, such that BQA (of J) plus $\underset{\smile}{AXB}$ forms a simple closed curve C containing in its interior the interior of J, and therefore $\underset{\smile}{ADB}$ and $\underset{\smile}{QP}+P$.

Since sides are preserved under T, there exists a simple closed curve C' in S' containing $B'Q'A'$ and whose interior contains $\underset{\smile}{A'D'B'}$ and $\underset{\smile}{Q'P'}+P'$. The curve C' is composed of the arcs $B'Q'A'$ of J', and $\underset{\smile}{B'X'A'}$ in the exterior of J'. The interior of C' therefore contains the interior of J', in fact $\underset{\smile}{A'D'B'}$ divides the interior of C' into two parts, the interior of $J'=A'D'B'Q'A'$ and the interior of $A'D'B'X'A'$. Since $\underset{\smile}{Q'P'}+P'$ is exterior to J' and interior to C', it must lie within $A'D'B'X'A'$. Since T is continuous, the limit point Q' of $\underset{\smile}{Q'P'}+P'$ must lie on or within $A'D'B'X'A'$, whereas the arc $\underset{\smile}{B'Q'A'}$ is exterior to $A'D'B'X'A'$. This is the desired contradiction.

Theorem II. *If M and M' are continuous curves lying in planes S and S' respectively, and if there exists a continuous* (1-1) *correspondence T, such that $T(M)=M'$, and such that sides are preserved under T, then there exists a continuous* (1-1) *correspondence U, such that $U(S)=S'$, and such that if, for any point P of M, $T(P)=P'$, then $U(P)=P'$.*

Proof. Since we have considered in Theorem I the case where M contains no simple closed curve, we shall assume here that M contains at least one simple closed curve. The proof of Theorem II follows, in the main, that of Theorem I. We shall go into detail only when the proof differs from that of Theorem I.

By Schoenflies's definition of a continuous curve,* $S-M$ consists of one unbounded domain and a countable set of bounded domains, the same being true for $S'-M'$. The outer boundary of each bounded domain is a simple closed curve J whose interior contains a countable set of maximal connected subsets of $M-J$ (which we shall call "trees"), each having one and only one limit point on the simple closed curve.

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* Report, p. 290 and p. 295.

If M is a continuous curve, and N is a closed and connected subset of M, we shall call a maximal connected subset of $M-N$ which has one and only one limit point in N a *tree with respect to* N or a *tree in* $M-N$. The limit point in N is called the *foot* of the tree. The number of trees is countable, and if N is the outer boundary of a domain complementary to M, then not more than a finite number of trees can be of diameter greater than any given positive number. In case M contains no simple closed curve this definition is equivalent to the one given previously.

If α is the boundary of D_1, the unbounded domain in $S-M$, then $T(\alpha)=\alpha'$ is the boundary of D_1', the unbounded domain in $S'-M'$. For suppose it were possible that for some point P of α, $T(P)=P'$ is not in β', the boundary of D_1'. Then P' is separated from any point Q' in D_1' by β', and P' and Q' are therefore separated by some simple closed curve J' which is a subset of β'.* The point Q' is not interior to J'; therefore P' is interior to J'. But P is in α and is therefore not interior to $T^{-1}(J')=J$ or any other simple closed curve in M, and in this case interiors (and, therefore, sides) have not been preserved under T, contrary to hypothesis. A similar contradiction is arrived at if we suppose that for some point P' of β', $T^{-1}(P')=P$ is not a point of α. Therefore $\beta'=\alpha'$.

We shall next show that if α is the boundary of D_2, a bounded domain in $S-M$, then $T(\alpha)=\alpha'$ is the boundary of a bounded domain in $S'-M'$, which we may call D_2'. The boundary α contains a simple closed curve J, and α' contains J'. If N denotes the points of M interior to J, N consists of a countable set of trees. Since N' is identical with the points of M' interior to J', any maximal connected subset of N' is a tree with respect to J', otherwise T is not (1-1) and continuous, and therefore N' also consists of a countable set of trees. Let these be $N_1', N_2', N_3', \cdots$. Let L_i' be N_i' plus its foot on J' plus any points of $S'-M'$ lying in a complementary domain of M' whose boundary is in N_i'. The sets $L_1', L_2', L_3', \cdots$ are closed and connected and have no points in common. Furthermore, only a finite number of these sets can be of diameter greater than any given positive number. Under these conditions, $L_1'+L_2'+L_3'+\cdots$ is not connected.† Therefore not every point of the interior of J' lies in $L_1'+L_2'+\cdots$. Let P' be a point of the interior of J' not in $L_1'+L_2'+\cdots$, and let D_2' be the domain in $S'-M'$

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* R. L. Moore, *Concerning continuous curves in the plane*, loc. cit., Theorem V.

† See abstract by J. R. Kline, Bulletin of the American Mathematical Society, vol. 31 (1925), p. 300.

containing P'. The outer boundary of D_2' is a simple closed curve K'. If K' were entirely interior to J' or had one point in common with J', then K', D_2', and P' would be points of one of the sets L_i', contrary to our selection of P'. If K' had an arc in common with J', but were not identical with J', K' would contain an arc interior to J' joining two points of J', and such an arc cannot exist in any of the trees N_1', N_2', $\cdots$. Therefore, K' is identical with J'. If we suppose that any point P of α is such that P' is not in the boundary of D_2', or that any point Q' of the boundary of D_2' is such that Q is not in α, we obtain a contradiction by the same argument as that used in the case of the boundary of the unbounded domain. Therefore the boundary of D_2' is α'.

We shall next discuss the definition of "sides are preserved under T" for the case where M is any continuous curve.

Let D be any domain in $S-M$, and A and B two points on its boundary β; then any arc AXB such that $\underline{AXB}$ is in D, divides D into two domains, D_1 and D_2, such that $D=D_1+D_2+\underline{AXB}$. The boundary of D_i $(i=1, 2)$ is composed of $\underline{AXB}$ and β_i, a subset of β. Evidently, $\beta=\beta_1+\beta_2$. Suppose that A and B are two points such that β_1 and β_2 are the same for any choice of the arc $\underline{AXB}$. This will be the case if each of the points A and B is a non-cut point of β, i. e., a point whose removal does not disconnect β. If in $\beta_i (i=1, 2)$ we draw the arc AYB (there is only one such arc), and draw in D an arc $\underline{AXB}$ forming a simple closed curve J with AYB, then the discussion under Theorem I of "sides are preserved under T" holds with slight modifications. Therefore, under the above conditions, we can add to our definition of "sides are preserved under T" that J' contains $A'B'$ but no other points of M', that $J'-A'B'$ lies in D' (where D' is the domain in $S'-M'$ whose boundary is $T(\beta)=\beta'$), and that the interior of J' contains N' and no other points of M'.

As in the proof of Theorem I, we can construct a circle C containing M in its interior I, and a circle C' containing M' in its interior I'. Let α denote the boundary of the unbounded domain in $S-M$. Since α is a non-dense continuous curve separating the plane, α contains a simple closed curve J.* Let A and B be two points of J which are not feet of trees in $\alpha-J$. We can join A to D, any point of C, by an arc in $(I-M)+A+D$, and we can join B to any other point E of C by an arc in $(I-M)+B+E-AD$. If D' and E' are arbitrary points on C' we can draw similar arcs $A'D'$ and $B'E'$. Each of the arcs AB of J forms with AD, BE, and the proper one of the arcs DE of C a simple closed curve, and these two simple closed curves C_1 and C_2 contain and enclose all points of α. Similarly in S' are two simple closed curves C_1' and C_2' such that the points contained and enclosed by C_i' $(i=1, 2)$ are the points corresponding to those contained and enclosed by C_i.

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* Report, pp. 295–296.

Let us select an arbitrary positive number ϵ. Let $D_1, D_2, \cdots, D_n$ be a finite set of domains complementary to M, and $D_1', D_2', \cdots, D_n'$ a set of domains complementary to M', such that (1) if α_i is the boundary of D_i $(i=1, 2, \cdots, n)$, then $T(\alpha_i)=\alpha_i'$ is the boundary of D_i' and (2) all domains complementary to either M or M' and of diameter greater than ϵ occur in one set or the other. In each of the bounded domains, such as D_2, select two points A and B on the outer boundary J_2, which are not feet of trees in α_2-J_2, and join A to B by an arc in D_2 save for its end points. If we join A' to B' by an arc in D_2' save for its end points, this arc divides the interior of J_2' into the interiors of two simple closed curves, such that the points of M' contained and enclosed by either one of them are the points corresponding to those contained and enclosed by the corresponding one of the simple closed curves in S which AB forms with J_2.

Let us now consider any one of the above simple closed curves K formed by an arc AXB in M and $\underline{AYB}$ in $S-M$, and its corresponding simple closed curve $K'=A'X'B'Y'A'$ in the plane S'. Let the points of M contained and enclosed by K be denoted by N. Then $T(N)=N'$ is the set contained and enclosed by K'. Suppose either $N-AXB$ or $N'-A'X'B'$ contains a tree of diameter greater than $\frac{1}{9}\epsilon$, and let this tree and its corresponding one be denoted by T and T'. For definiteness let us suppose that the diameter of T is greater than $\frac{1}{9}\epsilon$. Let the foot of T be F. Let G_1 denote some point of T which belongs to α, the boundary of the complementary domain of M in which $\underline{AYB}$ was drawn, and such that an arc FG_1 of α is of diameter greater than $\frac{1}{18}\epsilon$. If $\alpha-G_1$ is connected, we shall denote G_1 by G. If $\alpha-G_1$ is not connected, it consists of a countable set of trees. Let W denote the one which contains F. The set $\alpha-W$ is closed and connected. It has therefore a non-cut point G distinct from G_1. The sets $(\alpha-W-G)$ and $(W+G_1)$ are connected and have G_1 in common. Therefore their sum, $\alpha-G$, is connected, and G is a non-cut point of α. Moreover, any arc in α from F to G passes through G_1 and therefore α contains an arc FG of diameter greater than $\frac{1}{18}\epsilon$.

If H is any point of $\underline{AYB}$, H can be joined to G by an arc interior to K, save for H, and having only G in common with M. In S', a similar arc $H'G'$ can be drawn. The points of M on and interior to $GFAHG$ will have their corresponding points on and interior to $G'F'A'H'G'$. Continuing this process a finite number of times we eventually arrive at the point where none of the simple closed curves K and K' contain a tree of diameter greater than $\frac{1}{9}\epsilon$.

The choice of sets $K_1, K_2, \cdots, K'_{m-1}$, and the simple closed curves $C_1, C_2, \cdots, C'_{m-1}$ is made as in Theorem I without any modifications. The correspondence U is defined as in Theorem I for all points of $S-M$ and for all points which are boundary points of domains complementary to M. If there are any other points in M, we shall define U as being identical with T.

The correspondence U defined in this way is evidently (1-1), and for points of M, U and T are identical. It remains to be shown that U is continuous, i. e., if $P_1, P_2, \cdots$ is a set of points of S approaching P as a sequential limit, then $P'_1, P'_2, \cdots$ approach $T(P)=P'$ as a sequential limit. In case P is a point of $S-M$, all except a finite number of the points of $P_1+P_2+\cdots$ lie in the same domain as does P, and we have defined U for the complementary domains in such a way that $P'_1, P'_2, \cdots$ will approach P'. In case P is a point of M and an infinite number of points of $P_1+P_2+\cdots$ lie in M, then P' is a limit point of $P'_1+P'_2+\cdots$ because U is identical with T for points of M, and T is continuous. In case P is a point of M, and an infinite number of points of $P_1+P_2+\cdots$ lie in a single domain of $S-M$, then P is a boundary point of that domain, and again U has been defined in such a way that $P'_1, P'_2, \cdots$ have P' as a limit point. In case, finally, P is a point of M, and only a finite number of points of $P_1+P_2+\cdots$ lie in M or in any single domain of $S-M$, then we can pick out an infinite subsequence $Q_1, Q_2, \cdots$, such that each Q_i is a point of $S-M$, and no two points Q_i, Q_j lie in the same domain. Let us associate with each Q_i a point R_i of the boundary of the domain in which Q_i lies. For any given ϵ only a finite number of the points R_i can be selected in such a way that the distance between Q_i and R_i is greater than ϵ. Therefore the set of points $R_1, R_2, \cdots$ also approach P as a sequential limit. Similarly in the plane S', the two sequences $Q'_1, Q'_2, \cdots$ and $R'_1, R'_2, \cdots$ approach the same limit. Since $R'_1, R'_2 \cdots$ are points of M', their limit is P'. Therefore in each case, U is continuous.

The author wishes to express his thanks to Professor John Robert Kline for suggesting this problem and for aiding in its solution.

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ON EXTENDING A CONTINUOUS (1-1) CORRESPONDENCE (SECOND PAPER)*

BY

HARRY MERRILL GEHMAN

I. Introduction

In a previous paper† we have given a condition which is necessary and sufficient in order that it be possible to extend a continuous (1-1) correspondence of two plane continuous curves to a correspondence of their planes. In the present paper, we shall give a similar condition for the case of two plane point sets of a more general type. Theorem 2 of this paper includes as special cases the two theorems of our *First paper*.

Definition. By an *E-set*, we shall mean a closed and bounded plane point set, each component‡ of which is a continuous curve, not more than a finite number of them being of diameter greater than any given positive number.

II. Some properties of an E-set and of the domains complementary to an E-set

1. If M is an E-set, we can let $M=m_0+M_1+M_2+\cdots$, where m_0 is totally disconnected and consists of all those components of M which consist of a single point, and for each value of i, M_i is a component of M. Note that m_0 may consist of an uncountable infinity of components of M, but $M-m_0$ consists of at most a countable infinity of components of M.

2. Of the domains complementary to an E-set M lying in a plane S, one is unbounded and the remainder are bounded. Let D denote the unbounded domain of $S-M$, and for each component M_x of M, let D_x denote the unbounded domain of $S-M_x$. The set D is a subset of each of the sets D_x, and hence is a subset of their common part.

Let N be the point set consisting of the sum of the sets $S-D_x$ corresponding to all the components M_x of M. The set N is an E-set, each component

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* Presented to the Society, October 29, 1927; received by the editors in December, 1928.

† H. M. Gehman, *On extending a continuous (1-1) correspondence of two plane continuous curves to a correspondence of their planes*, these Transactions, vol. 28 (1926), pp. 252–265. We shall refer to this paper as *First paper*.

‡ A *component* or *maximal connected subset* of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M.

of which has just one complementary domain. The set N therefore has just one complementary domain,* which is the same as the common part of all the sets D_x. Now since M is a subset of N, the set $S-N$ is a subset of $S-M$, and since $S-N$ is unbounded, it must be a subset of D. As we have shown above that D is a subset of $S-N$ (the common part of the sets D_x), it follows that $D=S-N$.

The boundary of D is the sum of the boundaries of the domains D_x corresponding to those components M_x of M which do not lie in a bounded complementary domain of any other component of M.

3. Let d denote a bounded domain of $S-M$. If M_x is any component of M, the domain d is a subset of some domain of $S-M_x$. If d is a subset of a bounded domain of $S-M_x$, then the diameter of M_x is greater than or equal to the diameter of d. Hence it follows from the definition of an E-set, that in the case of a finite number of components of M, d is a subset of a bounded complementary domain; in the case of the remaining components of M, d is a subset of the unbounded complementary domain.

Let $M_1, M_2, \cdots, M_n$ denote the components of M such that for each component M_i, there exists a bounded domain d_i of $S-M_i$ which contains d. Each domain d_i has an outer boundary† consisting of a simple closed curve J_i. The curves $J_1, J_2, \cdots, J_n$ are mutually exclusive since they belong to different components of M, and since each of them encloses d, it follows that of any pair of them, one encloses the other. Hence one of them, say J_k, is interior to each of the other simple closed curves of the set.

It follows by §2, after an inversion of the plane about a circle lying in d, that the outer boundary of d is the simple closed curve J_k, and that the entire boundary of d consists of the boundary of d_k and the boundary of each of the unbounded domains D_x complementary to a component M_x of M which lies in d_k and does not lie in a bounded complementary domain of any other component of M which lies in d_k.

4. We have shown in §3 that corresponding uniquely to each bounded domain d of $S-M$ is a bounded domain d_k complementary to a component M_k of M, such that d_k contains d and has the same outer boundary. Given any positive number ϵ, not more than a finite number of components of M are of diameter greater than ϵ, and each of these has not more than a finite number of bounded complementary domains of diameter greater than ϵ.

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* H. M. Gehman, *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), pp. 553–568. See Lemma F, p. 558.

† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260.

Hence there are not more than a finite number of domains of $S-M$ which are of diameter greater the ϵ.

5. By the methods used in §2 and the Lemma referred to there, and by means of an inversion of the plane in the case of a bounded domain, it follows that if P is a point of M which is a boundary point of a domain of $S-M$, then P is *accessible* from that domain. That is, if Q is any point of the domain, there exists an arc QP every point of which except P is a point of the domain.

III. INTERIOR-PRESERVING CORRESPONDENCES

DEFINITION.[*] Let M and M' be E-sets lying in planes S and S' respectively, and let T be a continuous (1-1) correspondence such that $T(M)=M'$. Then we shall say that T *preserves interiors* if given any two simple closed curves J of M and J' of M' which are such that $T(J)=J'$ and which enclose respectively the subset K of M and the subset K' of M', then $T(K)=K'$.

THEOREM 1. *If M and M' are E-sets lying in planes S and S' respectively, and if T is a continuous* (1-1) *correspondence such that $T(M)=M'$ and such that T preserves interiors, then if B is the boundary of a domain D of $S-M$, the set $T(B)$ is the boundary of a domain D' of $S'-M'$.*

In the case of the unbounded domain of $S-M$, the proof given in *First paper*, page 262, second paragraph, holds without change.

In the case of a bounded domain of $S-M$, the proof given in *First paper*, page 262, third paragraph, holds with the following changes: N consists of a countable set of trees and a (possibly uncountable) collection of components of M. Using the notation of §1 of Part II, we can denote this collection by $m_0+M_1+M_2+\cdots$. If D_i is the unbounded domain of $S-M_i$, let K_i denote $S-D_i$; and if D_i' is the unbounded domain of $S'-M_i'$, let K_i' denote $S'-D_i'$. Each component of the set $H=(L_1+L_2+\cdots)+m_0+(K_1+K_2+\cdots)$ is either one of the sets L_i, a point of m_0, or one of the sets K_i; similarly, each component of $H'=(L_1'+L_2'+\cdots)+m_0'+(K_1'+K_2'+\cdots)$ is one of the sets L_i', a point of m_0', or one of the sets K_i'. Furthermore since T preserves interiors, if a component of H is a set L_a, a point P of m_0, or a set K_b, then one component of H' is the set L_a', the point $T(P)=P'$ of m_0', or the set K_b', and conversely. Hence, as in *First paper*, not every point of I' (the interior of J') is a point of H', and it follows that $I'-H'$ consists of a single component, which is the required domain of $S'-M'$.

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[*] *First paper*, p. 260.

IV. CORRESPONDENCES WHICH PRESERVE SIDES OF ARCS IN THE SAME SENSE

DEFINITIONS. Let M and M' be E-sets lying in planes S and S' respectively, and let T be a continuous (1-1) correspondence such that $T(M)=M'$. We shall say that T *preserves sides of arcs*, if given any simple closed curve J in S containing an arc AB of M, then there exists a simple closed curve J' in S' containing $T(AB)=A'B'$ and such that if J encloses the subset K of M, then J' encloses $T(K)=K'$; and also if given any simple closed curve J' in S' containing an arc $A'B'$ of M', then there exists a simple closed curve J in S containing $T^{-1}(A'B')=AB$ and such that if J' encloses the subset K' of M', then J encloses $T^{-1}(K')=K$.* We shall say that T *preserves sides of arcs in the same sense*, if T preserves sides of arcs, and if given any two simple closed curves J_1 and J_2 in S containing arcs A_1B_1 and A_2B_2 of M respectively, then the corresponding simple closed curves J_1' and J_2' in S' can be constructed in such a way that if X_1, Y_1, Z_1 are three points of A_1B_1 and X_2, Y_2, Z_2 are three points of A_2B_2, such that the sense† $X_1Y_1Z_1$ on J_1 is the same as the sense $X_2Y_2Z_2$ on J_2, then the sense $T(X_1)T(Y_1)T(Z_1)$ on J_1' is the same as the sense $T(X_2)T(Y_2)T(Z_2)$ on J_2'; and similarly for any two simple closed curves in S' containing arcs of M'.

THEOREM 2. *If M and M' are E-sets lying in planes S and S' respectively, and if T is a continuous* (1-1) *correspondence such that $T(M)=M'$ and such that T preserves interiors and preserves sides of arcs in the same sense, then T can be extended to a correspondence between the planes S and S', i.e., there exists a continuous* (1-1) *correspondence U, such that $U(S)=S'$, and such that for each point P of M, $U(P)=T(P)$.*

1. If every component of M is an arc or a point, and no interior point of an arc XY of M is a limit point of $M-XY$, then an arc in the plane S may be passed through M, and an arc in S' may be passed through M'.‡ If M and M' are subsets of arcs, it has been proved§ that *any* continuous (1-1) correspondence of M and M' can be extended to a correspondence of the

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* In *First paper*, p. 253, we said that *sides are preserved under T*, if T has the above property. It seemed better to say here that *sides of arcs are preserved*. See Part V of this paper.

† J. R. Kline, *A definition of sense on closed curves in non-metrical plane analysis situs*, Annals of Mathematics, vol. 19 (1918), pp. 185–200, and J. R. Kline, *Concerning sense on closed curves in non-metrical plane analysis situs*, Annals of Mathematics, vol. 21 (1919), pp. 113–119.

‡ R. L. Moore and J. R. Kline, *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, vol. 20 (1919), pp. 218–223.

§ R. L. Moore, *Conditions under which one of two given closed linear point sets may be thrown into the other one by a continuous transformation of a plane into itself*, American Journal of Mathematics, vol. 48 (1926), pp. 67–72.

planes, which proves Theorem 2 for sets of this type. The theorem just cited is a special case of Theorem 2, because if the E-sets M and M' are subsets of arcs, any continuous (1-1) correspondence of M and M' preserves interiors (vacuously) and preserves sides of arcs in the same sense.

2. In all other cases, we shall define, for convenience, a positive and a negative sense for points on simple closed curves in the planes S and S'.

If M contains any simple closed curves, we shall select an arbitrary simple closed curve J_1 of M and three arbitrary points A, B, C of J_1 and shall say that the sense XYZ on the simple closed curve J in the plane S is *positive*, if and only if it is the same as the sense ABC on J_1; otherwise the sense XYZ on J is *negative*. In the same way, in the plane S' we shall say that the sense $X'Y'Z'$ on the simple closed curve J' is *positive*, if and only if it is the same as the sense $T(A)T(B)T(C)$ on $T(J_1)$.

If M contains no simple closed curve, but contains an arc AB such that an interior point P of AB is a limit point of $M-AB$, then we shall construct an arbitrary simple closed curve J_1 containing the arc AB of M, and enclosing a subset K of $M-AB$ which has P for a limit point. Then we shall say that the sense XYZ on the simple closed curve J in the plane S is *positive*, if and only if it is the same as the sense APB on J_1. Since T preserves sides of arcs, there exists a simple closed curve J_1' in S' containing the arc $T(AB)$ and enclosing $T(K)$, and we shall define the sense $X'Y'Z'$ on the simple closed curve J' in the plane S' as *positive*, if and only if it is the same as the sense $T(A)T(P)T(B)$ on J_1'. With the help of Lemma A of *First paper* and Kline's definition of sense, it is easily seen that the positive sense in S' as defined thus is independent of the particular selection of the simple closed curve J_1'.

If in every application of the condition that T preserves sides of arcs in the same sense, we let J_1 be the fixed curve J_1 defined above, it follows that if three points of an arc of M which is a subset of the simple closed curve J_2, are in a certain sense (either positive or negative) on J_2, then the three points corresponding under T to the given points are in the same sense on the simple closed curve J_2', regardless of the particular selection of J_2'.

3. By Theorem 1, there exists a (1-1) correspondence between the domains of $S-M$ and those of $S'-M'$. We shall now show how to define the correspondence U for points of a pair of corresponding domains and their boundaries, this being done in such a way that for each point P of M, $U(P)=T(P)$.

For simplicity, we shall define U for points of the unbounded domains of $S-M$ and $S'-M'$, which we shall denote by D and D' respectively. A similar method suffices for a pair of corresponding bounded domains, after an inversion of the plane.

4. As in §2 of Part II, let $N=S-D$ and let $N'=S'-D'$. If each component of N is considered as an element and each point of $S-N$ is considered as an element, the set of all such elements is topologically equivalent to the set of all points in a plane Σ.* The set of points in Σ corresponding to the components of N is a closed and totally disconnected set Γ. Similarly, if the components of N' and the points of $S'-N'$ are considered as elements, the set of all such elements is equivalent to the set of all points of a plane Σ', and the points of Σ' corresponding to the components of N' form a closed and totally disconnected set Γ'.

Since T preserves interiors, there exists a (1-1) correspondence V between the components of N and the components of N' such that a point P of M lies in a component N_x of N if and only if $T(P)$ lies in the component $V(N_x)$ of N'. The correspondence V considered as a correspondence between the points of Γ and the points of Γ' is continuous and (1-1). We have shown above that a continuous (1-1) correspondence between two closed and totally disconnected sets such as Γ and Γ', can be extended to their planes; that is, there exists a continuous (1-1) correspondence W such that $W(\Sigma)=\Sigma'$, and such that for each point P of Γ, $W(P)=V(P)$.

The correspondence W, considered as a correspondence between S and S', is a continuous (1-1) correspondence between the points of $S-N$ and the points of $S'-N'$. Furthermore W is such that if J is any simple closed curve in $S-N$, then J encloses the component N_x of N if and only if the simple closed curve $W(J)$ lying in $S'-N'$ encloses the component $V(N_x)$ of N'. This is equivalent to the statement that J encloses the point P of M if and only if $W(J)$ encloses the point $T(P)$ of M'.

5. For any given positive number ϵ, we can enclose† each component N_x of N by a simple closed curve J_x which has no points in common with N and which is such that every point of J_x and every point which is interior to J_x is at a distance less than ϵ from a point of N_x. Since N has the Heine-Borel property, some finite subset of this collection of simple closed curves encloses all points of N. Hence‡ there exists a finite set of simple closed curves $J_1, J_2, \cdots, J_k$ enclosing all points of N, each curve being exterior to

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* See Theorem 25, p. 427, of the paper by R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428. See also H. M. Gehman, *Concerning acyclic continuous curves*, loc. cit., pp. 558–559.

† R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ R. L. Moore and J. R. Kline, loc. cit., Lemma 1, p. 219.

every other one of the set, and every point lying on or interior to any one of the curves being at a distance less than ϵ from some point of N.

Under the correspondence W, these simple closed curves correspond in S' to a set of simple closed curves $J_1', J_2', \cdots, J_k'$ which enclose all points of N' and are mutually exterior. If it happens that the distance from some point lying on or interior to one of these curves to any point of N' is greater that or equal to ϵ, we can construct in S' a finite set of simple closed curves $C_1', C_2', \cdots, C_h'$ which enclose all points of N', are mutually exterior, are such that every point lying on or interior to one of the curves is at a distance less than ϵ from some point of N', and are such that each of the curves is interior to one of the curves J_i'. On account of this last condition, the set of simple closed curves $C_1, C_2, \cdots, C_h$ in S, where $C_i = W^{-1}(C_i')$, has all the properties of the set $J_1, J_2, \cdots, J_k$.

6. We shall now define the correspondence U for points of S which are not interior to any of the curves $C_1, C_2, \cdots, C_h$ as any continuous (1-1) correspondence such that $U(S) = S'$, $U(C_i) = C_i'$ for $i = 1, 2, \cdots, h$, and if $X_iY_iZ_i$ are three points of one of the curves C_i, then the sense $X_iY_iZ_i$ on C_i is the same as the sense $U(X_i)U(Y_i)U(Z_i)$ on C_i, *sense* being positive or negative as defined in §2. Note that the correspondence W can always be defined so as to have this last property, in which case we can let $U(P) = W(P)$ for all points P of S which are not interior to any of the curves C_i.

We shall next define U for points of D and of the boundary of D which are interior to the simple closed curve C, where C denotes any one of the curves $C_1, C_2, \cdots, C_h$.

7. If each component of M interior to C is either an arc or a point, and if no interior point of an arc XY of M is a limit point of $M - XY$, then just as in §1, we can construct an arc interior to C which contains all points of M which are interior to C, and hence there exists a continuous (1-1) correspondence U_1 such that (1) $U_1(S) = S'$, (2) for each point P of C, $U_1(P) = U(P)$, and (3) for each point P of M interior to C, $U_1(P) = T(P)$. We shall then let $U(P) = U_1(P)$ for all points of S interior to C.

8. If the subset of M interior to C is not of the above type, let M_1 be a component of M which is interior to C and which either contains an arc XY such that some interior point of XY is a limit point of $M - XY$, or contains a simple closed curve. If M_1 contains no simple closed curves, let A and B be the end points of the maximal arc of M_1 that contains the arc XY. If M_1 contains a simple closed curve, the component of the boundary of D which is a subset of M_1 contains a simple closed curve J_1. In this case, let A and B

be two points of J_1 which are not feet of trees* with respect to J_1 of the component of the boundary of D which contains J_1.

In either case, let E and F be two points of C. We can construct two non-intersecting arcs AE and BF in $S-C-M+A+B+E+F$.† Let $E'=U(E)$, $F'=U(F)$, $A'=T(A)$, and $B'=T(B)$, and in $S'-C'-M'+A'+B'+E'+F'$ we can construct two non-intersecting arcs $A'E'$ and $B'F'$.

Let G and H be two points which separate E and F on C and let $G'=U(G)$ and $H'=U(H)$. There are two and only two domains of $S-(C+AE+BF+M)$ which are interior to C and have boundary points on C. The boundary of one of these domains R_1 consists of the arc EGF of C, the arcs AE and BF, and a subset of M_1 containing an arc AKB which with the other three arcs forms a simple closed curve which is the outer boundary of the domain R_1. The other domain R_2 is bounded by the arc EHF of C, the arcs AE and BF, and a subset of M_1 containing an arc ALB which with the other three arcs forms a simple closed curve which is the outer boundary of the domain R_2. We have chosen A and B in such a way that either AKB and ALB are identical, or AKB and ALB have no interior points in common. It is also true that the boundary of R_1 (and that of R_2) contains the same points of M_1 regardless of how the arcs AE and BF are constructed.

Similarly in S', there are two and only two domains of $S'-(C'+A'E'+B'F'+M')$ which are interior to C' and have boundary points on C'. If the domain whose boundary contains G' is denoted by R_1', then the outer boundary of R_1' consists of $A'E'+E'G'F'+B'F'$ and one of the arcs $A'K'B'$, $A'L'B'$; the outer boundary of the other domain R_2' consists of $A'E'+E'H'F'+B'F'$ and the other one of the arcs $A'K'B'$, $A'L'B'$. The arcs $A'K'B'$, $A'L'B'$ either are identical or have no interior points in common. In the latter case, since the arcs $A'K'B'$ and $A'L'B'$ form a simple closed curve, it follows that the outer boundary of R_1' contains $A'K'B'$ because the following senses are either all positive or all negative: AKB on $AKBLA$, EGF on C, EGF on $EGFBKAE$, $A'K'B'$ on $A'K'B'L'A'$, and $E'G'F'$ on C'. It is also true that the subset of M_1' that lies in the boundary of R_1' (and that of R_2') is the same, regardless of how the arcs $A'E'$ and $B'F'$ are constructed. Since T preserves interiors and preserves sides of arcs in the same sense, it follows that a point P of M_1 is a point of the boundary of R_i $(i=1, 2)$, if and only if $T(P)$ is a point of the boundary of R_i'.

However it may happen that for a particular choice of the arcs AE, BF, $A'E'$, $B'F'$, a component M_2 of M lies in R_1, while $T(M_2)=M_2'$ lies in

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* *First paper*, p. 262.

† Part II, §5.

R_2'. We shall now show how, keeping $A'E'$ and $B'F'$ fixed, a new pair of arcs AE and BF may be constructed so that a component M_2 of M lies in R_1 if and only if M_2' lies in R_1'.

9. Let us suppose that the arcs AE, BF, $A'E'$, $B'F'$ have been constructed. Let N_i' $(i=1, 2)$ denote the set consisting of all components of M' that lie in R_i', and let $N_i=T^{-1}(N_i')$. Each of the sets M_1+N_1, M_1+N_2, $M_1'+N_1'$, $M_1'+N_2'$ is closed.

We shall now show that if we denote by N_{1i} $(i=1, 2)$ the subset of N_1 that lies in R_i, then $N_{12}+A+B$ is closed. Since the set $M_1+N_1=M_1+N_{11}+N_{12}$ is closed, and no point of N_{11} is a limit point of N_{12}, it follows that M_1+N_{12} is closed. Since N_{12} is a subset of R_2 and $T(N_{12})$ is a subset of R_1', it follows that any point P of M_1 which is a limit point of N_{12} must be a boundary point of R_2, such that $T(P)$ is a boundary point of R_1'. If AKB and ALB have no interior points in common, A and B are the only points of M_1 which have this property. If $AKB=ALB$, every point of the arc has this property, but if an interior point of the arc AB of M_1 were a limit point of N_{12}, the correspondence T would not preserve sides of arcs in the same sense. Hence in both cases $N_{12}+A+B$ is closed.

Let us join H to L by an arc in R_2+H+L. This arc divides R_2 into two domains R_3 and R_4 whose boundaries contain the points A and B respectively. If we let N_{1i} $(i=3, 4)$ denote the subset of N_{12} which lies in R_i, then the sets $N_{13}+A$ and $N_{14}+B$ are closed.

Since A is a component of the closed set $N_{13}+A$, we can (by using Zoretti's theorem) let $N_{13}=K_1+K_2+\cdots$, where (1) K_i and K_j have no points in common, if $i\neq j$, (2) K_i and $N_{13}-K_i$ are closed, (3) $K_1+\cdots+K_i$ contains all components of N_{13} which contain points at a distance from A greater than $1/2^i$. Note that since T preserves interiors, N_{13} contains all components of M which lie in a bounded complementary domain of a component of N_{13}. Hence we can assume that the sets K_1, K_2, $\cdots$ also satisfy condition (4): if a component M_x of M lies in a bounded complementary domain of a component of K_i, then M_x is a component of K_i.

For each value of i, the sets K_i and $M-K_i$ are closed, and we can therefore, as in §5, enclose K_i by a finite number of mutually exterior simple closed curves J_{i1}, J_{i2}, $\cdots$, J_{in_i}, lying in R_3, of diameter less than $1/2^{i-2}$, such that the exterior of each one contains $M-K_i+J_{11}+\cdots+J_{1n_1}+J_{21}+\cdots+J_{i-1,n_{i-1}}$. For simplicity, we shall denote each of the curves J_{hk} just defined, by J_i, where $i=n_1+n_2+\cdots+n_{h-1}+k$. The point set consisting of $A+J_1+J_2+\cdots$ is closed; in fact it is an E-set. Also the set consisting of M, the simple closed curve C, the arcs AE, BF, HL, all the curves J_i, and any finite number of arcs, forms an E-set.

Let us next select for each value of i, a point Q_i of the curve J_i, and a point P_i whose distance from A is less than $1/2^i$, which is an interior point of the arc EA, and which is (for $i>1$) an interior point of the arc $P_{i-1}A$. We can then construct an arc P_iQ_i in $P_i+Q_i+R_3-(P_1Q_1+\cdots+P_{i-1}Q_{i-1})-(J_1+J_2+\cdots)$ of diameter less than $1/2^{h-2}$, where h is determined by the equation $i=n_1+\cdots+n_{h-1}+k$ given above. The set consisting of A and all the arcs P_iQ_i also forms an E-set.

No point of a given arc P_iQ_i is a limit point of M or of any subset of $\theta_i=(J_1+\cdots+J_{i-1}+J_{i+1}+\cdots)+(P_1Q_1+\cdots+P_{i-1}Q_{i-1}+P_{i+1}Q_{i+1}+\cdots)$. We can therefore enclose P_iQ_i by a simple closed curve C_i in R_1+R_3 whose exterior contains $M+\theta_i+C_1+\cdots+C_{i-1}$ and which is of diameter less than $1/2^{h-2}$, the relation between i and h being given above. The curve C_i contains two arcs $P_{i1}Q_{i1}, P_{i2}Q_{i2}$ such that (1) P_{i1} and P_{i2} are interior points of the arc EA lying between P_{i-1} and P_i and between P_i and P_{i+1} respectively, (2) Q_{i1} and Q_{i2} are points of J_i such that the sense $Q_{i1}Q_iQ_{i2}$ on J_i is the same as the sense EGF on $EGFBKAE$, (3) the interior of the simple closed curve formed by these two arcs and the arcs $P_{i1}P_{i2}$ of AE and $Q_{i1}Q_iQ_{i2}$ is a subset of R_3-M and contains no points interior to J_i.

The set consisting of $AE-(P_{11}P_{12}+P_{21}P_{22}+\cdots)+P_{11}Q_{11}+P_{21}Q_{21}+\cdots+P_{12}Q_{12}+P_{22}Q_{22}+\cdots+(J_1-Q_{11}Q_1Q_{12})+(J_2-Q_{21}Q_2Q_{22})+\cdots$ is an arc* from A to E which we may denote by $(AE)_1$. By our method of construction, the domain R_1 determined by using $(AE)_1$ in place of AE, contains $N_{11}+N_{13}$, while the distribution of points of N_2 between the domains R_1 and R_2 has not been changed. In the same way a new arc from B to F, which we may denote by $(BF)_1$, can be constructed so that the domain R_1 determined by $(AE)_1$ and $(BF)_1$ contains $N_{11}+N_{13}+N_{14}=N_1$. If any points of N_2 lie in the domain R_1 determined by $(AE)_1$ and $(BF)_1$, we can in the same way determine new arcs $(AE)_2$ and $(BF)_2$, such that the domain R_1 determined by them contains N_1, and the domain R_2 contains N_2.

10. Then as in *First paper*, pp. 264–265 and pp. 256–258, we can obtain a finite set of points $G_1=E, G_2, \cdots, G_m=F, \cdots, G_n$ lying in the order $G_1G_2\cdots G_m\cdots G_nG_1$ on the simple closed curve C, and a set of points $H_1=A, H_2, \cdots, H_m=B, \cdots, H_n$ of M_1, and a set of arcs $G_1H_1, \cdots, G_nH_n$, such that (1) the arc G_iH_i has no interior points in common with $C+M$, (2) the simple closed curve C_i formed by the arcs G_iH_i, $G_{i+1}H_{i+1}$, the arc G_iG_{i+1} of C that contains none of the points $G_1, \cdots, G_{i-1}, G_{i+2}, \cdots, G_n$, and an arc H_iH_{i+1} of the boundary of D, encloses no tree of $M_1-H_iH_{i+1}$

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* This may be established with the aid of Theorem 2 of H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, vol. 27 (1926), pp. 381–384.

which is of diameter greater than $\epsilon/9$, or whose corresponding set under T is of diameter greater than $\epsilon/9$, and (3) a point P of M is interior to C_i if and only if $T(P)$ is interior to the simple closed curve C_i' formed by the arcs $U(G_iG_{i+1})=G_i'G_{i+1}'$ of C', the arc $T(H_iH_{i+1})=H_i'H_{i+1}'$ of M_1, and arcs $G_i'H_i'$, $G_{i+1}'H_{i+1}'$ lying in $S'-C'-M'+G_i'+H_i'+G_{i+1}'+H_{i+1}'$.

Then by using the method of §5 and the method of *First paper*, p. 265 and pp. 258–259, we can express the points of D and its boundary which are interior to C as the sum of a finite number of domains containing no points of M, a finite number of domains of diameter less than ϵ, and the boundaries of these domains, the same being true for points of D' and its boundary which are interior to C'. By the method of §9, a point P of M lies in a domain in S, if and only if $T(P)$ lies in the corresponding domain in S'. The correspondence U is then defined as in *First paper*, p. 265 and pp. 259–260, so as to satisfy the conclusion of Theorem 2.

V. LOGICAL RELATIONS BETWEEN VARIOUS KINDS OF CORRESPONDENCES

DEFINITIONS. Let M and M' be E-sets lying in planes S and S' respectively, and let T be a continuous (1-1) correspondence such that $T(M)=M'$. We shall say that T *preserves sides*, if given any simple closed curve J in S containing a connected* subset L of M, then there exists a simple closed curve J' in S' containing $T(L)=L'$ and such that if J encloses the subset K of M, then J' encloses $T(K)=K'$; and similarly for any simple closed curve J' in S' containing a connected subset L' of M'. We shall say that T *preserves sides in the same sense*, if T preserves sides, and if given any two simple closed curves J_1 and J_2 in S containing connected subsets L_1 and L_2 of M respectively, then the corresponding simple closed curves J_1' and J_2' in S' can be constructed in such a way that if X_1, Y_1, Z_1 are three points of L_1 and X_2, Y_2, Z_2 are three points of L_2, such that the sense $X_1Y_1Z_1$ on J_1 is the same as the sense $X_2Y_2Z_2$ on J_2, then the sense $T(X_1)T(Y_1)T(Z_1)$ on J_1' is the same as the sense $T(X_2)T(Y_2)T(Z_2)$ on J_2'; and similarly for any two simple closed curves in S' containing connected subsets of M'.

1. The following seven theorems, which follow from the definitions, give the complete set of logical relations connecting the five kinds of correspondences which have been defined in this paper.

THEOREM 3. *If T preserves sides in the same sense, then T preserves sides.*

THEOREM 4. *If T preserves sides of arcs in the same sense, then T preserves sides of arcs.*

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* Since M is closed, L is necessarily either an arc or the curve J.

THEOREM 5. *If T preserves sides, then T preserves sides of arcs.*

THEOREM 6. *If T preserves sides in the same sense, then T preserves sides of arcs in the same sense.*

THEOREM 7. *If T preserves sides, then T preserves interiors.*

THEOREM 8. *If T preserves interiors and preserves sides of arcs, then T preserves sides.*

THEOREM 9. *If T preserves interiors and preserves sides of arcs in the same sense, then T preserves sides in the same sense.*

2. On account of Theorems 3, 6, 7, and 9, we can make a purely formal change in the statement of Theorem 2 by replacing "T preserves interiors and preserves sides of arcs in the same sense" by "T preserves sides in the same sense." However the conclusion of Theorem 2 no longer remains true if we replace this phrase by either "T preserves sides" or "T preserves sides of arcs in the same sense."

3. If the E-set M is connected, i.e., if M is a continuous curve, then it follows from the theorems of *First paper*, that the conditions (1) T preserves sides in the same sense, (2) T preserves sides of arcs in the same sense, (3) T preserves sides, (4) T preserves sides of arcs, are logically equivalent. Each of them implies that T preserves interiors but not conversely.*

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* *First paper*, Lemma B, p. 261. Using the definitions of the present paper, Lemma B is *If M is connected and T preserves sides of arcs, then T preserves interiors.*

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CONCERNING THE CUT POINTS OF CONTINUA*

BY

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In this paper propositions will be established concerning the collection of all the cut points of a given plane† continuum. It will be shown that there does not exist an uncountable collection of mutually exclusive subcontinua of a given continuum M each of which contains at least one cut point of M. With the aid of this fundamental theorem it is shown, among other things, that the set of all the cut points of a continuum M which are irregular points of M is necessarily countable and, indeed, that all save a countable number of the cut points of any continuum M are points of Menger order two of M.

1. PRELIMINARY NOTIONS. INTERVALS AND SEGMENTS OF A CONTINUUM

Definitions. The points, or point sets, A and B of a continuum M are said to be separated in M by the point X of M provided that $M-X$ is the sum of two mutually separated point sets $M_a(X)$ and $M_b(X)$ containing A and B respectively. The notation $K(A, B)$ will be used to denote the set of all points of M which separate A and B in M.

Let A and B be any two points of a continuum M and X and Y any two distinct points of $K(A,B)$. Then X will be said to *precede* Y on M in the order from A to B provided that X belongs to the set $M_a(Y)$; X will be said to follow Y provided that X belongs to the set $M_b(Y)$. It is easily seen that if X precedes Y, then Y follows X. If X precedes Y, then by the *interval of M from X to Y* (or from Y to X), or the *interval XY* (or YX) *of M*, will be meant the set of points $[M_b(X)+X]\cdot[M_a(Y)+Y]$, and by the *segment* of M from X to Y will be meant the set of points $M_b(X)\cdot M_a(Y)$. If X follows Y, then since Y precedes X, by the interval XY of M is meant the set of points $[M_b(Y)+Y]\cdot[M_a(X)+X]$, and the segment XY (or YX) of M means the set of points $M_a(X)\cdot M_b(Y)$. The notation $I(X, Y)$ for the interval of M from X to Y and $S(X, Y)$ for the segment of M from X to Y will be used in this paper. Clearly $S(X, Y)=I(X, Y)-(X+Y)$.

The following two propositions can be proved with little difficulty from the above definitions.

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* Presented to the Society, September 9, 1927; received by the editors in September, 1927.

† Quite recently I have established some very general results on the cuttings of continua, as a consequence of which it follows that all the theorems in this paper hold true in a euclidean space of n dimensions, and nearly all of them hold in any separable metric space which is locally compact. See my forthcoming paper *Concerning collections of cuttings of connected point sets.*

THEOREM 1. *Let A and B be any two points of a continuum M, and let X and Y be any two distinct points of $K(A, B)$, where X precedes Y. Then M is the sum of three continua $M_a(X)+X$, $I(X, Y)$, and $M_b(Y)+Y$, where $[M_a(X)+X]\cdot I(X, Y)=X$, $[M_b(Y)+Y]\cdot I(X, Y)=Y$, and $[M_a(X)+X]\cdot[M_b(Y)+Y]=0$.*

THEOREM 2. *Using the same notation as in Theorem 1, suppose that each of the sets $M_a(X)$, $M_b(X)$, $M_a(Y)$, and $M_b(Y)$ is connected. Then if $S(X, Y)$ contains a point of $K(A, B)$, $S(X, Y)$ is connected.*

2. THE PRINCIPAL RESULTS

THEOREM 3. *If H is any uncountable subset of the set G of all the cut points of a continuum M, then there exist two points A and B of H which are separated in M by uncountably many of the points of H.*[*]

There exists a countable subset D of M such that every point of M is a limit point of D. Let F be the collection of all possible pairs of points of D. Then F is countable. For each pair of points u, v of F, let $K_h(u, v)$ denote the set of all points of H that separate u and v in M. Then $H=\sum_F K_h(u, v)$. For let X be a point of H. Then since X is a cut point of M, $M-X=M_1+M_2$, where M_1 and M_2 are mutually separated point sets. Since every point of M is a limit point of D, it follows that M_1 contains a point U of D and M_2 contains a point V of D. Then X separates U and V in M, and hence it belongs to $K_h(U, V)$. Therefore $H=\sum_F K_h(u, v)$; and since F is countable and H is uncountable, it follows that there exists at least one pair P, Q of F such that $K_h(P, Q)$ is uncountable.

Let E denote the set of points $K_h(P, Q)$. Then since E is uncountable, it contains a point A which is a point of condensation of E, i.e., every vicinity of A contains uncountably many points of E. Now for at least one of the sets $M_p(A)$ and $M_q(A)$, say for $M_q(A)$, it is true that A is a point of condensation of the subset of E which belongs to $M_q(A)$. Let B be a point of E belonging to $M_q(A)$. Then by Theorem 1 it readily follows that the segment AB of M must contain an uncountable subset E_0 of E. Let X be any point of E_0. Then since X belongs to $S(A, B)$, it follows by Theorem 1 that neither of the continua $M_p(A)+A$ and $M_q(B)+B$ contains the point X. And as these continua contain the points P and Q respectively, therefore they are subsets of $M_p(X)$ and $M_q(X)$ respectively. Hence A and B are separated in M by each point X of E_0, and as E_0 is an uncountable subset of H, the truth of Theorem 3 is therefore established.

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[*] The referee has kindly called my attention to the fact that Theorems 3 and 4 of this paper are valid in any number of dimensions, since my proofs of these theorems make no use of the dimensionality of the space in which the point sets lie.

THEOREM 4. *Let G denote the set of all the cut points of a continuum M. Then for all, save possibly a countable number, of points X of G it is true that $M-X$ is the sum of two mutually separated and connected point sets.*[*]

Suppose, on the contrary, that there exists an uncountable subset D of G such that if X is any point of D, then $M-X=M_1(X)+M_2(X)$, where $M_1(X)$ and $M_2(X)$ are mutually separated point sets and at least one of them is not connected. Then by Theorem 3 there exist two points A and B of D which are separated in M by each point of an uncountable subset H of the set D. For each point X of H, $M-X=H_a(X)+H_b(X)+H_0(X)$, where $H_a(X)$, $H_b(X)$, and $H_0(X)$ are mutually separated point sets, and $H_a(X)$ and $H_b(X)$ contain A and B respectively.

Now for each point X of H, the set H, and indeed the set $K(A, B)$, is a subset of the set of points $H_a(X)+H_b(X)+X$, because this set of points is connected and contains both A and B and therefore must contain every point of M which separates A and B in M. It follows that if X and Y are distinct points of H, the set of points $H_0(Y)$ belongs to the set $H_a(X)+H_b(X)$, and the set $H_0(X)$ belongs to the set $H_a(Y)+H_b(Y)$; and therefore the sets $H_0(X)$ and $H_0(Y)$ are mutually exclusive. By the Zermelo postulate there exists a set of points E such that for each point X of H there is exactly one point P_x of E which belongs to the set of points $H_0(X)$, and for each point P_x of E there is exactly one point X of H such that the set $H_0(X)$ contains the point P_x. Since H is uncountable, E must be uncountable. Hence there exists a point P_z of E which is a limit point of E. But there exists a point Z in H such that P_z belongs to the set $H_0(Z)$, and $E-P_z$ belongs to the set of points $H_a(Z)+H_b(Z)$. Clearly this is impossible, because $H_a(Z)$, $H_b(Z)$, and $H_0(Z)$ are mutually separated point sets. Thus the supposition that Theorem 4 is not true leads to a contradiction.

Theorem 4 can also be proved very readily with the aid of a statement given, without proof, by C. Zarankiewicz in his paper *Sur les points de division dans les ensembles connexes.*[†]

THEOREM 5. *Let M be any continuum whatever. Then there does not exist an uncountable collection of mutually exclusive subcontinua*[‡] *of M each of which contains at least one cut point of M.*

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[*] Since this paper was submitted to the editors, a result somewhat more general than Theorem 4 has been published by Kuratowski and Zarankiewicz; cf. *A theorem on connected point sets*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 571–575.

[†] Cf. Fundamenta Mathematicae, vol. 9, pp. 124–171, footnote to p. 137.

[‡] In this paper the hypothesis that a set is a continuum is understood to imply that the set contains more than one point.

Before proceeding with the proof of Theorem 5, I shall establish two lemmas which will be needed in the proof.

LEMMA 1. *If C is any subcontinuum of a continuum M, then not more than a countable number of the bounded maximal connected subsets of $M-C$ can contain cut points of M.*

Suppose, on the contrary, that there exists an uncountable collection G of bounded maximal connected subsets of $M-C$ each of which contains at least one cut point of M. From each element g of G select exactly one cut point P_g of M, and let K denote the set of all points $[P_g]$ thus selected. For each point P_g of K, $M-P_g=M_c(P_g)+M_0(P_g)$, where $M_c(P_g)$ and $M_0(P_g)$ are mutually separated sets and $M_c(P_g)$ contains the continuum C. Let H denote the sum of all the point sets of the collection G. Then since each element of G has* at least one limit point in C, it follows that for each point P_g of K, the set of points $H-g$ belongs to the set $M_c(P_g)$. By the Zermelo postulate, there exists a set of points E such that for each point P_g of K there is exactly one point Q_g of E which belongs to the set of points $M_0(P_g)$, and for each point Q_g of E there is exactly one point P_g of K such that the set of points $M_0(P_g)$ contains the point Q_g. It is readily seen that no two of the sets $[M_0(P_g)]$ can have a common point. Therefore, as H is uncountable, both K and E must be uncountable. Hence E contains a point Q_j which is a limit point of E. But clearly this is impossible, because there exists a point P_j in K such that Q_j belongs to the set $M_0(P_j)$ and $E-Q_j$ belongs to the set $M_c(P_j)$, and $M_0(P_j)$ and $M_c(P_j)$ are mutually separated sets by definition. This contradiction proves Lemma 1.

LEMMA 2. *If M is any continuum, C is any simple closed curve, and G_0 denotes the collection of all the maximal connected subsets of M contained in C plus its interior, then not more than a countable number of elements of G_0 can contain cut points of M which lie within C.*

Suppose, on the contrary, that G_0 contains an uncountable subcollection G each element g of which contains at least one cut point P_g of M lying within C.

(1) Not more than a countable number of elements g of G can have the property that $M-P_g$ can be expressed as the sum of two mutually separated sets one of which lies wholly within C. For suppose the contrary is true. Then, clearly, for each such element g of G, the point P_g is a cut point of the continuum $M+C$. But since each such point P_g lies on a bounded maximal

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* See my paper *Concerning point sets which can be made connected by the addition of a simple continuous arc*, these Transactions, vol. 29 (1927), footnote to p. 754.

connected subset of $(M+C)-C$, and no two different points $[P_g]$ can lie on the same maximal connected subset of $(M+C)-C$, this is contradictory to Lemma 1.

(2) Let G_1 denote the collection of all those elements g of G such that however $M-P_g$ be expressed as the sum of two mutually separated point sets $M_1(P_g)$ and $M_2(P_g)$, each of these sets must contain at least one point of C. It follows by (1) that G_1 must be uncountable. It is clear that if g is any element of G_1 and we let $g_1(P)=g\cdot M_1(P_g)$ and $g_2(P)=g\cdot M_2(P_g)$, then $g-P_g=g_1(P)+g_2(P)$, and $g_1(P)$ and $g_2(P)$ are mutually separated point sets each of which contains at least one point on C. Now C is not a subset of M, for if it were, then G_0 would have only one element. Let the maximal segments of $C-M\cdot C$ be denoted by $S_1, S_2, S_3, \cdots$. Let g be any element of G_1. Denote the end points of S_1 by A and B, and let AOB be the arc of C from A to B which does not contain S_1. On AOB, in the order from A to B, let X_g be the first point belonging to g. Then X_g belongs either to $g_1(P)$ or to $g_2(P)$, say to $g_1(P)$. Then on the arc X_gB of AOB, in the order from X_g to B, let Y_g be the first point belonging to $g_2(P)$. It is easy to see that the segment X_gY_g of the arc AOB contains at least one segment S_i of the sequence $S_1, S_2, S_3, \cdots$. Let A_i and B_i denote the end points of S_i, where A_i precedes B_i on AOB in the order from A to B. Then if T_1 and T_2 denote the arcs AA_i and B_iB respectively of AOB, $g_1(P)$ has at least one point on one of these arcs and $g_2(P)$ has at least one point on the other. Thus it is seen that for each element g of G_1 there exist two segments of the sequence $S_1, S_2, S_3, \cdots$ such that $g_1(P)$ has a point on one of the complementary arcs of C of these two segments and $g_2(P)$ has a point on the other. Then since the number of segments $S_1, S_2, S_3, \cdots$ is countable, and G_1 is uncountable, there exist two segments S_k and S_j of the sequence $S_1, S_2, S_3, \cdots$ such that if T and L denote the complementary arcs of C, then for uncountably many elements g of G_1 it is true that $g_1(P)$ has at least one point on one of the arcs T and L, and $g_2(P)$ has at least one point on the other. Let G_2 denote the collection of all those elements of G_1 which have this property.

(3) There are not more than a countable number of elements g of G_2 such that $T+L$ contains as many as three points of g. For suppose the contrary is true. Then there exists an uncountable subcollection E of G_2 such that each element e of E has at least two points on one of the arcs T and L, say on T, and at least one point on L. Let A and B denote the end points of T, where A is an end point of S_k, and for each element e of E, let A_e and B_e respectively denote the first and last points on T in the order from A to B which belong to e. Then for no two elements e and f of E can the segments

A_eB_e and A_fB_f of T have a common point. For suppose they do. Then either the segment A_eB_e contains a point of f or the segment A_fB_f contains a point of e. The two cases are alike, so let us suppose that the segment A_eB_e contains a point X of f. Now L contains at least one point Y of f. There exists an arc XZY from X to Y which lies, except for the points X and Y, wholly without C. It is readily seen that the segments S_k and S_j belong to different complementary domains R_k and R_j respectively of the continuum $f+XZY$. Since A_e can be joined to A and B_e to B by arcs lying, except for their end points, in the exterior of C and containing no point whatever of XZY, it follows that A_e must belong to R_k and B_e to R_j. Then since e is connected, it must have at least one point in common with $f+XZY$. But as the segment XZY lies without C, e can have no point in common with it. Hence e must have a point in common with f. This is contradictory to the fact that e and f are different maximal connected subsets of M lying in C plus its interior. Therefore no two segments A_eB_e and A_fB_f can have a common point. But since E is uncountable, then the number of these segments must be uncountable. Clearly this is impossible. This contradiction proves the statement made in the first sentence of this paragraph.

(4) It follows from (3) that there exists an uncountable subcollection G_3 of elements of G_2 such that if e is any element of G_3, then $e_1(P)$ has exactly one point X_e on T and $e_2(P)$ has exactly one point Y_e on L. But since G_3 is uncountable, it contains an element g which contains the sequential limiting set H of some infinite sequence V of elements of G_3. Then H is connected and contains at least one point on each of the arcs T and L. But since g has in common with $T+L$ only the points X_g and Y_g, H must contain both X_g and Y_g. Now some infinite subsequence V_1 of V has the property that all of its elements belong to one of the sets $M_1(P_g)$ and $M_2(P_g)$, say to $M_1(P_g)$. But H is also the limiting set of the sequence V_1, and the point Y_g belongs to the set $M_2(P_g)$. Hence $M_2(P_g)$ contains a limit point of $M_1(P_g)$, contrary to the fact that $M_1(P_g)$ and $M_2(P_g)$ are mutually separated point sets. Thus the supposition that Lemma 2 is not true leads to a contradiction.

Proof of Theorem 5. Suppose, contrary to Theorem 5, that there exists an uncountable collection G of mutually exclusive subcontinua of M each of which contains at least one cut point of M. From each element g of G select exactly one cut point P_g of M, and let N denote the set of all points $[P_g]$ thus selected. Since N is uncountable, there exists a square S which encloses uncountably many points of N. Let N_1 be the set of all those points of N which are within S. There exists a positive number d and an uncountable subset N_2 of N_1 such that if G_2 denotes the collection of all those elements of G which contain a point of N_2, then each element of G_2 is of diameter greater than $4d$. Let us divide S plus its interior into a finite collection E of squares plus their interiors by lines parallel and perpendicular to the bases of S in such a way that the diameter of each square of E is less than d. Then either some side T of one of the squares of E contains uncountably many points of N_2, or else some square of E encloses uncountably many of these points. In either case it is readily seen that there exists a rectangle R_1 of diameter less than $2d$ which encloses uncountably many of the points of N_2. There exists a rectangle R_2 concentric with and within R_1 which also encloses an uncountable subset N_3 of N_2.

Let F denote the collection of all the maximal connected subsets of M contained in R_2 plus its interior. Since N_3 is uncountable and each point of N_3 belongs to some element of F, and since, by Lemma 2, not more than a countable number of elements of F can contain points of N_3, it follows that some element K of F must contain an uncountable subset N_4 of N_3. It follows by a theorem of R. L. Moore's* that uncountably many points of N_4 must be cut points also of the continuum K. Let N_5 be the set of all those points of N_4 which are cut points of K. By Theorem 3, there exist two points A and B of N_5 which are separated in K and also in M by uncountably many points of N_6. Let N_6 be the set of all those points of N_5 which separate A and B in K and also in M, and let G_6 be the collection of all those elements of G_2 which contain a point of N_6. Since each element of G_6 is of diameter greater than $4d$, and R_1 is of diameter less than $2d$, each element of G_6 must contain a point of R_1 and also a point of R_2. For each element g of G_6, the maximal connected subset H_{0_g} of g which contains the point P_g and lies within R_1 has at least one limit point A_{P_g} which belongs to R_1. Let H_{P_g} denote the continuum obtained by adding to H_{0_g} all of its limit points. Now since A and B belong to N_5, the elements g_a and g_b of G_2 which contain A and B respectively contain subcontinua H_a and H_b respectively which contain A and B respectively and at least one point U and V respectively on R_1 but which lie wholly in R_1 plus its interior. It follows by Theorem 4 that N_6 contains an uncountable subset N_7 such that if X is any point of N_7, then $M-X=M_a(X)+M_b(X)$, where $M_a(X)$ and $M_b(X)$ are mutually separated and connected point sets containing A and B respectively, and also $K-X=K_a(X)+K_b(X)$, where $K_a(X)$ and $K_b(X)$ are connected sets belonging to $M_a(X)$ and $M_b(X)$ respectively. Let E_0 denote the collection of all the continua $[H_x]$ selected above which correspond to points X of N_7. At least one of the arcs of R_1 from U to V must contain uncountably many points

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* R. L. Moore, *Concerning the cut points of continuous curves and of other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

A_x which correspond to points X of N_7 and belong to continua H_x of the collection E_0; let UOV denote one which does, and let N_8 be the set of all those points of N_7 such that the corresponding point A_x belongs to the arc UOV.

Now if X and Y are two points of N_8 such that X precedes Y on K in the order from A to B (see §1), I shall show that the point A_x precedes the point A_y on the arc UOV in the order from U to V. Suppose, on the contrary, that A_y precedes A_x on UOV in the order from U to V. Then U and A_x separate the points V and A_y on R_1, and if UWA_x is an arc from U to A_x which lies, except for its end points, in the exterior of R_1, then it is easy to see that the points A_y and V lie in different complementary domains of the continuum $UWA_x+H_a+K_a(X)+H_x$. Then since the continuum $H_b+K_b(Y)+H_y$ contains both A_y and V, therefore it must contain at least one point in common with $UWA_x+H_a+K_a(X)+H_x$. This is impossible, for no one of the sets H_b, $K_b(Y)$, and H_y can have a point in common with any one of the sets UWA_x, H_a, $K_a(X)$, and H_x. Thus the supposition that A_x does not precede A_y on the arc UOV in the order from U to V leads to a contradiction.

Now since N_8 is uncountable, the set of corresponding points $[A_x]$ is uncountable. Accordingly there exists one of these points A_z which is a limit point of a subset S_1 of the remaining ones belonging to the arc UA_z of UOV and also of a subset S_2 of the remaining ones belonging to the arc A_zV of UOV. Now the point A_z belongs either to $M_a(Z)$ or to $M_b(Z)$, say to $M_a(Z)$. But this is impossible, since A_z is a limit point of S_2, and if A_x is any point of S_2, it was shown above that the point X follows the point Z on M in the order from A to B (for the order on M and on K is the same), i.e., X belongs to $M_b(Z)$; and thus A_z is a limit point of $M_b(Z)$, which is absurd. Likewise we arrive at a contradiction if we suppose that A_z belongs to $M_b(Z)$. Thus the supposition that Theorem 5 is not true leads to a contradiction and the theorem is therefore established.

THEOREM 6. *If E is any subset of the set of all the cut points of a continuum M, then there are not more than a countable number of points X of E such that X belongs to some subcontinuum of M which contains no other point of E.*

Suppose, on the contrary, that there exists an uncountable subset D of E such that each point X of D belongs to some subcontinuum N_x of M containing no other point of E. Then by Theorems 3 and 4 it follows that there exist two points A and B of M and an uncountable subset F of D such that if X is any point of F, X separates A and B in M and furthermore each of the sets $M_a(X)$ and $M_b(X)$ is connected.

Now for not more than a countable number of points X of F is it true that there exists some other point Y of F such that the segment XY of M contains no point whatever of F. For let Q be the set of all points X of F such that there exists a point Y of F such that $S(X, Y)$ contains no point of F, and suppose, contrary to the statement just made, that Q is uncountable. It follows that there exists an uncountable collection of the segments $[S(X, Y)]$ no two of which are identical and no one of which contains a point of F. Let $S(X_1, Y_1)$ and $S(X_2, Y_2)$ be any two of these segments, where X_1 precedes Y_1 and X_2 precedes Y_2 on M in the order from A to B. Then $S(X_1, Y_1)$ and $S(X_2, Y_2)$ are mutually exclusive. For suppose they have a point P in common. One of these segments is not a subset of the other one, for they are not identical. Suppose $S(X_2, Y_2)$ is not a subset of $S(X_1, Y_1)$. Let K denote the set of points common to $S(X_1, Y_1)$ and $S(X_2, Y_2)$. Now not both of the points X_1 and Y_1 can be limit points of K; for if they were, they would both belong to $I(X_2, Y_2)$ and at least one of them would have to belong to $S(X_2, Y_2)$, contrary to the fact that $S(X_2, Y_2)$ contains no point of F. The two cases are alike, so let us suppose X_1 is not a limit point of K. Then it is easy to see that K is closed except possibly for the point Y_1. Since K belongs to $S(X_1, Y_1)$, it must belong to $M_a(Y_1)$. But $M_a(Y_1)$ is connected; and since $M_a(Y_1)-K$ contains no limit point of K, therefore K must contain at least one limit point of $M_a(Y_1)-K$. But clearly this is impossible, since by Theorem 1 it follows that $S(X_1, Y_1)$ and $S(X_2, Y_2)$ are open subsets of M. Thus the supposition that $S(X_1, Y_1)$ and $S(X_2, Y_2)$ are not mutually exclusive leads to a contradiction. Hence no two of the segments of the collection $[S(X, Y)]$ can have a common point. But since, by Theorem 1, each of these segments is an open subset of M and since $[S(X, Y)]$ is an uncountable collection, this clearly is impossible.* Thus the supposition that Q is uncountable leads to a contradiction.

Now let H denote the set of points $F-Q$. Then since F is uncountable and Q is countable, H must be uncountable. By hypothesis, for each point X of H there exists a subcontinuum N_x of M which contains X but which contains no other point whatever of E. Now if X and Y are any two distinct points of H, the continua N_x and N_y can have no point in common. For suppose they do have a point P in common. The point Y belongs either to $M_a(X)$ or to $M_b(X)$, say to $M_b(X)$. Since X and Y belong to H, the segment XY of M must contain at least one point O of F, and O separates A and B in M. But the set of points $M_a(X)+M_b(Y)+N_x+N_y$ is connected and contains both A and B and does not contain the point O. Clearly this is impossible. Thus the supposition that N_x and N_y have a point in common leads to a contradiction. Now since H is uncountable, the collection of

* One would have to contain a limit of the others.

continua $[N_x]$ must be uncountable. But no two of these continua have a common point, and each of them contains at least one cut point of M. This is contradictory to Theorem 5. Thus the supposition that Theorem 6 is not true leads to a contradiction, and the theorem is proved.

3. THE ORDER AND REGULARITY OF THE CUT POINTS OF A CONTINUUM

Definitions. The point P of a continuum M is said to be a point of *Menger order* n *of* M* provided that for each positive number ϵ there exists a domain U containing P and of diameter less than ϵ and whose boundary has not more than n points in common with M, and furthermore n is the smallest positive integer such that this property is preserved. A point P of a continuum M will be called a regular or an irregular point of M according as M is or is not connected im kleinen at P.

THEOREM 7. *Let G denote the set of all the cut points of any continuum M. Then all save possibly a countable number of the points of G are points of Menger order two of M.*

Suppose, on the contrary, that there exists an uncountable subset H of G no point of which is a point of order two of M. Then by Theorem 3, there exist two points A and B of H which are separated in M by each point of an uncountable subset D of H. It follows by Theorem 6 that there exists at least one point P of D such that every subcontinuum of M which contains P must contain at least one point of D distinct from P. It is easy to see that each of the sets $M_a(P)$ and $M_b(P)$ must be connected and that P is a limit point of a subset D_1 of D belonging to $M_a(P)$ and also of a subset D_2 of D belonging to $M_b(P)$.

Let K_1 denote the continuum $M_a(P)+P$ and K_2 the continuum $M_b(P)+P$. There exists a sequence of points $X_1, X_2, X_3, \cdots$, belonging to D_1 and having P as its sequential limit point and such that for each positive integer n, X_n precedes X_{n+1} on M in the order from A to B. Now let us consider the sequence of intervals $I(X_1, X_2)$, $I(X_2, X_3)$, $I(X_3, X_4), \cdots$, of M. R. G. Lubben has shown† that if N denotes the sequential limiting set of this sequence, then N exists and is closed and connected. Clearly N contains the point P. It must consist of P alone. For suppose it does not. Then N is a subcontinuum of K_1 which contains P. It is readily seen that N cannot contain any point of the set of points

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* Cf. K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 272–306.

† See an abstract of his paper *Concerning limiting sets* in the Bulletin of the American Mathematical Society, vol. 32 (1926), p. 14. See also Zarankiewicz, loc. cit.

$$L = M_a(X_1) + \sum_{i=1}^{\infty} I(X_i, X_{i+1}).$$

But L is connected and P is a limit point of L. Hence K_2+L is connected and contains both A and B but contains in common with N only the point P. Hence N contains P but contains no other point of D. This contradicts our definition of the point P. Thus it follows that N is identical with P.

Now let ϵ be any positive number. Then since the sequential limiting set of the sequence of intervals $I(X_1, X_2)$, $I(X_2, X_3)$, $\cdots$, consists of just the point P, and since P is the sequential limit point of the sequence of points $X_1, X_2, X_3, \cdots$, it is readily seen that there exists a positive integer k such that if I denotes the set of points $\sum_{i=k-1}^{\infty} I(X_i, X_{i+1})$, every point of I is at a distance less than $\epsilon/4$ from P. Now by Theorem 1, $I(X_k, P)$ is connected, and it contains X_k. It does not contain X_{k-1}, because X_{k-1} precedes X_k on M in the order from A to B. Hence $I(X_k, P)$ must be a subset of $M_b(X_{k-1})$, and since it is a subset also of K_1, it must be a subset of $I+P$. But $I+P$ is of diameter less than $\epsilon/2$. Hence $I(X_k, P)$ is of diameter less than $\epsilon/2$. In an entirely similar manner it is shown that D_2 contains a point Y_k such that $I(P, Y_k)$ is of diameter less than $\epsilon/2$. Then $I(X_k, Y_k)$ contains P and is of diameter less than ϵ. And since the maximal connected subset of $S(X_k, Y_k)$ which contains P must have both X_k and Y_k as limit points, it is easily seen that there exists a domain of diameter less than ϵ which contains P and whose boundary has in common with M just the points X_k and Y_k. Hence P is a point of Menger order two of M. But by supposition P belongs to H, and no point of H is a point of Menger order two of M. Thus the supposition that Theorem 7 is not true leads to a contradiction.

THEOREM 8. *If A and B are any two points of a continuum M, and P is any point belonging to $K(A, B)$ and having the property that every subcontinuum of M which contains P contains at least one point of $K(A, B)$ distinct from P, then P is a point of Menger order two of M.*

The argument given to prove Theorem 7 also proves Theorem 8.

THEOREM 9. *If P is any cut point of a continuum M which is a point of Menger order two of M, then $M-P=M_1(P)+M_2(P)$, where $M_1(P)$ and $M_2(P)$ are connected point sets and P is a point of order one (i.e., an end point in the Menger sense) of each of the continua $M_1(P)+P$ and $M_2(P)+P$.*

THEOREM 10. *If P is any cut point of a bounded continuum M which is a point of Menger order two of M, then P belongs to the boundary of just one complementary domain of M.*

By a theorem of R. L. Moore's*, P belongs to the boundary of at least one complementary domain of M. Suppose, contrary to this theorem, that P belongs to the boundary of two complementary domains R_1 and R_2 of M. Then there exists† a simple closed curve J which encloses P and is of diameter less than half the diameter of R_1 and also less than half the diameter of R_2, and which has in common with M just two points A and B. Then of the two arcs of J from A to B, one of them, say AXB, must belong to R_1, and the other, AYB, must belong to R_2. But from Theorem 9 and a theorem of R. L. Moore's‡ it follows that there exists a simple closed curve C enclosing one of the sets $M_1(P)$ and $M_2(P)$ and not the other (where $M-P=M_1(P)+M_2(P)$, as in Theorem 9), and containing in common with M only the point P. It is easy to see that C must contain at least one point U of AXB and at least one point V on AYB. Then if t denotes the arc of M from U to V which does not contain P, then t contains no point of M. But this is impossible, because U and V belong to different complementary domains of M. Thus the supposition that Theorem 10 is not true leads to a contradiction.

THEOREM 11. *If G denotes the set of all the cut points of a continuum M, then all save possibly a countable number of the points of G are regular points of M.*

Theorem 11 is a direct consequence of Theorem 7 and of Menger's§ theorem that a continuum M is connected im kleinen at each of its points which is a point of finite order of M.

THEOREM 12. *If D denotes the collection of all the cut points $[P]$ of a continuum M such that P is an irregular point of some subcontinuum of M, then D is countable.*

Theorem 12 follows at once from Theorem 7 and the fact that any point P of Menger order two of a continuum M is a regular point of every subcontinuum of M which contains P.

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* R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213.

† This follows readily from a theorem of R. G. Lubben's. See the abstract of his paper in the Bulletin of the American Mathematical Society, vol. 32 (1926), p. 114.

‡ R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11(1925), pp. 469–476, Theorem 2. See also R. G. Lubben, loc. cit.

§ Cf. K. Menger, loc. cit.

4. Miscellaneous results

In this section some theorems will be stated with little or no proof given. These theorems, in general, either follow readily from the above propositions already proved or else they can be proved readily using methods similar to those employed in the above proofs.

Theorem 13. *Let G denote the set of all the cut points of any continuum M. Then G is a subset of the sum of a countable number of bounded subcontinua of M each of which is irreducible between some pair of points.*

Theorem 13 can be proved with the aid of Lemma 2 to Theorem 5 and methods similar to those used in the proof of Theorem 4, together with the theorem of Janiszewski's* that every two points of a bounded continuum M lie together in a subcontinuum of M which is irreducible between those two points.

Theorem 14. *The set G of all the cut points of a continuous curve M (bounded or not) is a subset of the sum of a countable number of simple continuous arcs which belong to M.*

Theorem 15. *If N is any subcontinuum of a continuum M which is irreducible between some pair of points A and B of M, then all save possibly a countable number of the cut points of M which belong to N must separate A and B in M.*

Corollary. *If t is any simple continuous arc belonging to a continuous curve M, and A and B denote the end points of t, then all save possibly a countable number of the cut points of M which lie on t separate A and B in M.*

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* S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), vol. 16 (1912).

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NON-SEPARATED CUTTINGS OF CONNECTED POINT SETS

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Reprinted from the
Transactions of the Mathematical Society
Vol. 33, No. 2, pp. 444-454

NON-SEPARATED CUTTINGS OF CONNECTED POINT SETS*

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1. We shall consider a connected, metric and separable space which we denote by M. A subset X of M is called a cutting of M provided that the complement $M-X$ of X is not connected and hence is the sum of two mutually separated sets $M_1(X)$ and $M_2(X)$; X is said to separate two points or point sets A and B in M when the sets $M_1(X)$ and $M_2(X)$ can be so chosen that $M_1(X) \supset A$ and $M_2(X) \supset B$, and is said to separate a single set N in M when $M_1(X)$ and $M_2(X)$ can be chosen so that $N \cdot M_1(X) \neq 0 \neq N \cdot M_2(X)$.

A collection G of subsets of M will be called *non-separated* provided that the elements of G are mutually exclusive and no element of G separates any other element of G in M.

A subset P of M is said to have the potential order α in M relative to a given collection G of subsets of M provided that α is the least cardinal number such that there exists a monotonic decreasing sequence $[U_i]$ of neighborhoods of P such that $P = \prod_1^\infty \overline{U}_i$ and such that for each i, the boundary $F(U_i)$ of U_i is a subset of the sum of α of the sets of the collection G.

In this paper we shall show, first, that if G is any uncountable non-separated collection of cuttings of M then *all save a countable number of the elements of G have the potential order 2 in M relative to G.* Now obviously if the elements of any collection G of mutually exclusive cuttings of M are connected or if they reduce to single points, then the collection G is non-separated. And since for the case where M is compact, the potential order of a point of M is the same as its order in the Menger-Urysohn sense, our theorem yields as corollaries many important known results concerning the cut points and connected cuttings of connected sets and of continua; for example: (1) the theorem of Wazewski-Menger‡ that the ramification points of any acyclic continuous curve are countable, (2) the theorem of Kuratowski and Zarankiewicz§ that the set of all points of any connected set M whose complement in M is neither connected nor the sum of two connected point sets is countable; (3) the theorem of the author* that all save a countable number of the cut points of any continuum are points of order 2 of M in the Menger-Urysohn sense; and (4) other results concerning cuttings due to Zarankiewicz† and to the author.‡

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* Presented to the Society, February 22, 1930; received by the editors July 5, 1930.

† Fellow, John Simon Guggenheim Memorial Foundation.

‡ See Wazewski, Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 49; and Menger, Fundamenta Mathematicae, vol. 10 (1927), p. 108.

§ Bulletin of the American Mathematical Society, vol. 33 (1927), p. 571.

Second, with the aid of this theorem we shall show that if the space M contains an uncountable non-separated collection G of cuttings, then there exists an upper semi-continuous collection S of elements such that all save a countable number of the sets of G are elements of S and such that every two elements of S may be separated in M by some third element. In case M is compact, the decomposition space S is an acyclic continuous curve.

Finally, we shall prove an existence theorem to the effect that *every locally connected space M contains an uncountable non-separated collection of cuttings.* Therefore, the above mentioned decomposition is always realisable for locally connected sets M, and notably for the case where M is a continuous curve, this decomposition gives rise to a decomposition space which is a non-degenerate acyclic continuous curve.

2. **Preliminary lemmas.** Let X and Y be any two cuttings of M and set

$$\text{(i)} \qquad M - X = M_1(X) + M_2(X),$$

$$\text{(ii)} \qquad M - Y = M_1(Y) + M_2(Y),$$

representing decompositions of $M-X$ and $M-Y$ respectively into mutually separated sets. Then if i, j, r, and s are positive integers such that $i+j=3=r+s$, it follows immediately that the following equation is valid:

$$(2.1) \qquad M = M_i(X) + M_r(Y) + M_j(X)\cdot M_s(Y) + X + Y.$$

With the aid of this equation, we deduce at once the result

(2.2) *If neither of the sets X and Y separates the other, we may choose the indices i and r such that*

$$\text{(a)} \qquad X \subset M_r(Y) \text{ and } Y \subset M_i(X);$$

and these relations imply also the relations

$$\text{(b)} \qquad M_j(X)\cdot M_s(Y) = 0,\ M_j(X) + X \subset M_r(Y), \text{ and } M_s(Y) + Y \subset M_i(X).$$

Clearly this is the case, because by virtue of the relations (a) we may omit the last two terms in equation (2.1); and since M is connected, the term $M_j(X)\cdot M_s(Y)$ must vanish. This fact gives at once the remaining two relations (b).

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* These Transactions, vol. 30 (1928), p. 606.

† See Fundamenta Mathematicae, vol. 12 (1928), pp. 119–125.

‡ See Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 87–104.

Now let G be any non-separated collection of cuttings of M and let $E(a, b)$ be the collection of all those elements of G which separate two given points a and b in M. Let X and Y be any two elements of $E(a, b)$ and let the indices in (i) and (ii) be chosen so that

$$\text{(iii)} \qquad M_1(X)\cdot M_1(Y) \supset a \text{ and } M_2(X)\cdot M_2(Y) \supset b.$$

The element X of $E(a, b)$ will be said to precede the element Y, and this fact is indicated by the notation $X<Y$, provided that for at least one set of decompositions satisfying (i), (ii) and (iii) it is true that $X \subset M_1(Y)$. We shall now show that this definition gives a natural order to the elements of $E(a, b)$.

First, for any two elements X and Y of $E(a, b)$, at least one of the relations $X<Y$ and $Y<X$ must be valid. For if X does not precede Y, then by (2.2), (a), $r=2$ and hence $s=1$. By (b) and (iii) it follows that $j=2$ and hence $i=1$. Therefore by (a), $Y \subset M_1(X)$, which means $Y<X$.

Second, only one of the relations $X<Y$ and $Y<X$ can be valid. For if $X<Y$, then [for any set of decompositions whatever satisfying (i), (ii), (iii)], in (2.2), $r=1$ and hence $s=2$. By (b) and (iii) it follows that $j=1$ and hence $i=2$. Therefore by (a), $Y \subset M_2(X)$, which is incompatible with $Y<X$.

Finally, for any three elements Z, X and Y of $E(a, b)$, the relations $Z<X$, $X<Y$ imply that $Z<Y$. For then $Z \subset M_1(X)$ and $X \subset M_1(Y)$. Hence in (2.2), $r=1$ and $s=2$. By (b) and (iii) it follows that $j=1$. Therefore by the second relation in (b), $Z \subset M_1(X)+X \subset M_1(Y)$, which gives $Z<Y$.

Thus we have proved the following result:

(2.3) *If each element of the non-separated collection $E(a, b)$ of subsets of M separates the two points a and b in M, then the collection $E(a, b)$ possesses a natural order.*

For convenience we give here a lemma concerning ordered sets due to Zarankiewicz* which will be used below.

LEMMA (Zarankiewicz). *If K is any ordered subset of M, then the set H of all points p of K such that p is not at the same time a limit point of the set P_p of all points of K preceding p and also of the set F_p of all points of K following p is countable.*

The space M being separable and metric, it therefore contains a countable sequence $R_1, R_2, R_3, \cdots$ of open sets which is equivalent to the set of all open subsets of M. Now let H_1 be the set of all points of K which are not limit points of their predecessors, and let $H_2=H-H_1$. For each point p of H_1 let $n(p)$ be the least positive integer such that $R_{n(p)}$ contains p but contains no point of K which precedes p. Then if p and q are distinct points of H_1 and

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* See Fundamenta Mathematicae, vol. 12 (1928), p. 119.

$p<q$, then since $R_{n(q)}$ does not contain p, it follows that $n(p)\neq n(q)$, and therefore H_1 is countable. A similar argument proves H_2 countable; and hence H is countable.

3. THEOREM. *If G is any uncountable non-separated collection of cuttings of a connected, metric, and separable space M, then all save possibly a countable number of the elements of G have the potential order 2 in M relative to G.*

Suppose, on the contrary, that G contains an uncountable subcollection G_1 no element of which has the potential order 2 in M relative to G. Now there exist two points a and b of M such that the collection $E(a, b)$ of all those elements of G_1 which separate a and b in M is uncountable; for M being separable, there exists a countable subset D of M such that $\bar{D}=M$; and since every element of G_1 which contains no point of D must separate some pair of points of D in M, and since the set of all pairs of points of D is countable, it follows that for at least one pair of points a, b of D, the set $E(a, b)$ is uncountable.

By §2 the elements of the collection $E(a, b)$ possess a natural order; and if K is a point set which contains exactly one point x of each element X of $E(a, b)$ and contains no other points, then K is an ordered point set. Indeed for each pair x, y of points of K, set $x<y$ provided that $X<Y$. By the Zarankiewicz lemma, the set H of all points p of K which are not at the same time a limit point both of their predecessors and of their successors is countable. Let $H(a, b)$ be the collection of all those sets X of $E(a, b)$ such that the corresponding point x in K belongs to $K-H$. Then $H(a, b)$ is uncountable and each element X of $H(a, b)$ contains a point x which is a limit point of the sum of the predecessors of X and also of the sum of the successors of X.

Now for each element X of $H(a, b)$, there exist mutually separated sets $M_1(X)$ and $M_2(X)$ such that

$$M - X = M_1(X) + M_2(X),\ M_1(X) \supset a \text{ and } M_2(X) \supset b.$$

And with the aid of what has just been shown it follows immediately that there exist two infinite sequences of elements $X_1, X_2, X_3, \cdots$ and $Y_1, Y_2, Y_3, \cdots$ of $H(a, b)$ such that, for each n,

$$(1)\qquad X_n < X_{n+1} < X < Y_{n+1} < Y_n,$$

and such that X contains a point which is a limit point both of $\sum X_n$ and $\sum Y_n$.

Since by supposition no element of $H(a, b)$ can have the potential order 2 in M relative to G, it follows that if for each element X of $H(a,b)$, $V_n(X)$ denotes the set of points $M-[M_1(X_n)+M_2(Y_n)]$, then there exists at least one point p_x belonging to the point set

$$\prod_1^\infty V_n(X) - X,$$

for if this were not the case, then by virtue of (1) and equation (2.1) in which substitute X_n for X, Y_n for Y, 1 for i and 2 for r, it follows that $V_n(X) \supset M_2(X_n)\cdot M_1(Y_n) \supset X$; and if for each point p of $M_2(X_n)\cdot M_1(Y_n)$ we take a neighborhood N_p of p of diameter less than 1/4 the distance from p to the set of points $\overline{M_1(X_n)}+\overline{M_2Y({}_n)}$, and call $U_n(X)$ the sum of all the neighborhoods N_p, then it follows readily that

$$X \subset M_2(X_n)\cdot M_1(Y_n) \subset U_n(X) \subset \overline{U_n(X)} \subset V_n(X);$$

and hence $F[U_n(X)] \subset X_n+Y_n$, $U_n(X) \subset U_{n-1}(X)$ and $X=\prod_1^\infty \overline{U_n(X)}$; but then X has the potential order 2 in M relative to G, contrary to supposition.

Now if X and Y are any two elements of $H(a, b)$, $X \neq Y$, it follows that $p_x \neq p_y$. For suppose $X<Y$. Then since X contains a limit point of the sum of its successors in $E(a, b)$ but contains no limit point of $M_2(Y)$, it follows that there exist two elements Y_k and Y_m in the "Y-sequence" in (1) for the element X such that

$$X < Y_k < Y_m < Y;$$

and since Y contains a limit point of the sum of its predecessors in $E(a, b)$ but contains no limit point of $M_1(Y_m)$, it follows that there exists an element X_n of the "X-sequence" for Y in (1) such that

$$X < Y_k < Y_m < X_n < Y.$$

Consequently it follows with the aid of (2.2) that

$$p_x \subset M_1(Y_k) + Y_k \subset M_1(Y_m)$$

and

$$p_y \subset M_2(X_n) + X_n \subset M_2(Y_m),$$

and hence $p_x \neq p_y$.

Now let L denote the set of all points $[p_x]$ for all elements X of $H(a, b)$. Then L is uncountable and is an ordered set; indeed, it is only necessary to set $p_x<p_y$ when $X<Y$. Therefore by the Zarankiewicz lemma, there exists a point p_x of L which is a limit point both of its predecessors and of its followers, and hence both of $\sum X_n$ and of $\sum Y_n$, where the sequences $[X_n]$ and $[Y_n]$ satisfy (1). But $\sum X_n \subset M_1(X)$ and $\sum Y_n \subset M_2(X)$; and p_x must then belong either to $M_1(X)$ or to $M_2(X)$ and be a limit point of the other, contrary to the fact that these two sets are mutually separated. Thus the supposition that our theorem is false leads to a contradiction.

4. Consequences of §3. Let G be any uncountable non-separated collection of cuttings of M. Then since the product of any family $[\overline{U}_n]$ of closed sets is closed, §3 yields at once the result

(α) *All save a countable number of the elements of G are closed point sets.*

Now if X is any element of G such that $M-X$ is not the sum of two connected point sets, X cannot have a potential order 2 in M relative to G. For $M-X=M_1(X)+M_2(X)+M_3(X)$, where the sets $M_1(X)$, $M_2(X)$, and $M_3(X)$ are mutually separated and contain points a_1, a_2 and a_3 respectively; and if X had the potential order 2 relative to G, there would exist two elements X_1 and X_2 of G and a neighborhood U of X such that $F(U)\subset X_1+X_2$, $X_1\subset M_1(X)$, $X_2\subset M_2(X)$ and $\overline{U}\cdot(a_1+a_2+a_3)=0$; but then it would readily follow that the point set $M_3(X)\cdot(M-\overline{U})$ is non-vacuous and is both open and closed, contrary to the fact that M is connected. Thus in consequence of the theorem in §3 we have

(β) *The complement of each element of G, with the exception of a countable number of such elements, consists of exactly two components.*

Let us denote by ρ the property of any subset N of M not to be separated in M by any single element of G. Clearly each element X of G has the property ρ. We shall now show that

(γ) *All save a countable number of the elements of G are saturated in M relative to the property ρ.*

If, on the contrary, G contains an uncountable subcollection G_1 no element of which is saturated relative to the property ρ, then for each element Z of G_1 there exists at least one point p_z which is not separated from Z in M by any single element of G. Under these conditions it follows by the theorem and proof in §3 that there exist two points a and b of M and three elements Z, X and Y of $E(a, b)$ (the collection of all those elements of G_1 which separate a and b) such that $X<Z<Y$, and $M_2(X)\cdot M_1(Y)$ contains Z but does not contain the point p_z and also such that $X+Y$ does not contain p_z. But then by equation (2.1) we have either $p_z\subset M_1(X)$ or $p_z\subset M_2(Y)$. This is impossible because in the first case X separates p_z and Z in M and in the second case Y separates p_z and Z in M.

A cutting X of M is said to be an irreducible cutting of M provided that no proper subset of X is a cutting of M.

(δ) *All save a countable number of the elements of G are irreducible cuttings of M.*

If this is not so, there exists an uncountable collection G^0 of cuttings of M such that for each element X^0 of G^0 there exists an element X of G and a point p_x of X such that $X^0\subset X-p_x$. Since G is non-separated, it follows at once that G^0 is non-separated. Therefore by (γ) there exists an element X^0

of G^0 which is saturated relative to the property ρ defined by the collection G^0. Consequently there exists an element Y^0 of G^0 which separates X^0 and p_x in M, and one has $M-Y^0=M_1(Y^0)+M_2(Y^0)$, where $M_1(Y^0)\supset X^0$ and $M_2(Y^0)\supset p_x$. But then $M-Y=M_1(Y^0)\cdot(M-Y)+M_2(Y^0)\cdot(M-Y)$, and thus Y separates X in M (for $Y\cdot(X^0+p_x)\subset Y\cdot X=0$), which contradicts the non-separatedness of G.

We prove now the following general theorem:

THEOREM. *Every uncountable non-separated collection G of cuttings of a connected, metric, and separable space M contains a subcollection Q which contains all save possibly a countable number of the elements of G and such that each element X of Q has the following properties*: (a) *X is closed*; (b) *$M-X$ is the sum of two mutually separated connected point sets*; (c) *X is saturated in M relative to the property ρ defined by the collection Q, i.e., for every point p of $M-X$, there exists an element Y of Q which separates X and p in M*; (d) *X is an irreducible cutting of M*; *and* (e) *X has the potential order 2 in M relative to Q.*

To obtain the collection Q, let D be a countable subset of M which is dense in M and let us omit from G: (1) every element which does not possess each of the properties (a), (b), and (d); (2) every element which separates in M some pair of points a, b of D which are separated by only a countable number of elements of G; (3) every element which separates some pair a, b of points of D and contains no point p having the property that every neighborhood of p contains points of uncountably many distinct elements of G which separate a and b. Let G_1 denote the collection of the elements of G remaining after these omissions. Then by virtue of (α), (β) and (δ), together with the facts that there are only a countable number of pairs of points of D and that in the space M every uncountable set of points contains a point of condensation of itself, it follows that G_1 contains all save possibly a countable number of the elements of G.

Now let us omit from G_1 every element which is not saturated in M relative to the property ρ defined by the collection G_1 and also every element which does not have the potential order 2 in M relative to G_1. Let Q be the collection of elements of G_1 remaining after these omissions. Then Q contains all save a countable number of the elements of G_1 and hence also of G, and every element X of Q has the desired properties (a)-(e). Clearly X has properties (a), (b) and (d), for every element of G_1 has these properties. It remains to show that X has properties (c) and (e).

To show that X has property (c), let p be any point of $M-X$. There exists an element Y of G_1 which separates X and p, because every element of Q is saturated in M relative to the property ρ defined by G_1. Hence $M-Y$

$=M_1(Y)+M_2(Y)$, where $M_1(Y)\supset X$ and $M_2(Y)\supset p$. Also $M-X=M_1(X)+M_2(X)$, where $M_2(X)\supset Y$. Thus if a and b are points of $M_1(X)$ and $M_2(Y)$ respectively belonging to D, both X and Y separate a and b in M, and we have $X<Y$ in the order from a to b. Now there exists also an element Z of G_1 which separates X and Y in M, and it follows from §2 that Z also separates a and b in M, and we have the order $X<Z<Y$. Thus $Z\subset M_2(X)\cdot M_1(Y)$. Since X and Y are closed, $M_2(X)\cdot M_1(Y)$ is a neighborhood of Z, and hence it contains points of (and therefore contains all of) uncountably many elements of G which separate a and b in M. Therefore there exists at least one of these elements, say W, which belongs to Q, for all but a countable number of the elements of G belong to Q. Thus we have the order $X<W<Y$; and since $p\subset M_2(Y)$, it follows that W separates X and p in M. Consequently X has property (c).

Since X has the potential order 2 in M relative to G_1, there exist, as shown in §3, two points a and b of M such that X belongs to the collection $E(a, b)$ of all those elements of G_1 which separate a and b in M and such that there exist two sequences $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ of elements of $E(a, b)$ so that

$$X_n < X_{n+1} < X < Y_{n+1} < Y_n$$

and such that if $U_n=M_2(X_n)\cdot M_1(Y_n)$, then $X=\prod_1^\infty \overline{U}_n$. Now for each n there exist, by virtue of property (c), two elements X_n' and Y_n' of Q belonging to $E(a, b)$ such that $X_n<X_n'<X<Y_n'<Y_n$. Hence if U_n' denotes the point set $M_2(X_n')\cdot M_1(Y_n')$, one has $U_n'\subset U_n$. Hence $X=\prod_1^\infty \overline{U}_n'$, and since $F(U_n')\subset X_n'+Y_n'$ and since clearly the sequence $[U_n]$ contains an infinite subsequence $[U_{n_i}]$ such that $U_{n_{i+1}}\subset U_{n_i}$, it follows that X has the potential order 2 in M relative to Q. This completes the proof.

5. Decomposition of M by means of a non-separated collection G every element of which is saturated relative to property ρ. Let G be any non-separated collection of subsets of M each of which is saturated in M relative to the property ρ defined by G. For each point e of M which belongs to no element of G, let E denote the point set consisting of e together with all points p of M which are not separated in M from e by any single element of G. Let S denote the collection whose elements are the elements of G together with all such point sets E thus defined. Clearly each element of S is closed and every point of M belongs to exactly one element of S. We shall show next that the collection S is non-separated.

Suppose, on the contrary, that some element X of S separates some pair of points p and q belonging to an element Y of S. Then $M-X=M_1(X)+M_2(X)$, where $M_1(X)\supset p$ and $M_2(X)\supset q$. Now by virtue of the definition of the collections G and S, it follows that there exists an element Z of G which separates X and p in M. Hence $M-Z=M_1(Z)+M_2(Z)$, where $M_1(Z)\supset X$ and $M_2(Z)\supset p$. Since Z belongs to G, it cannot separate Y in M; and therefore $p+q\subset Y\subset M_2(Z)$. But then

$$M - Z = [M_1(Z) + M_1(X)\cdot M_2(Z)] + M_2(X)\cdot M_2(Z),$$

and we have a separation of $M-Z$ into two mutually separated sets containing the points p and q respectively of Y, contrary to the fact that since Z belongs to G it cannot separate Y in M. Therefore S is non-separated.

Now clearly every element of S is saturated in M relative to the property ρ defined by the collection S. Consequently every two elements X and Y of S are separated in M by some third element of S. With the aid of this property it follows immediately that the collection S is upper semi-continuous,* i.e., there does not exist a sequence $X_1, X_2, X_3, \cdots$ of elements of S and two sequences $[p_i]$ and $[q_i]$ of points such that $p_i+q_i\subset X_i$ and which have sequential limit points p and q respectively belonging to two different elements P and Q respectively of S. For there exists an element X of S such that $M-X=M_1(X)+M_2(X)$ where $M_1(X)\supset P$ and $M_2(X)\supset Q$; and since for each i, X_i is a subset either of $M_1(X)$ or of $M_2(X)$, either $M_1(X)$ or $M_2(X)$ contains X_i for infinitely many i's; but this is impossible, for both p and q are limit points of every infinite subsequence of $X_1, X_2, X_3, \cdots$.

Now in case the space M is compact, the elements of S are closed and compact, and if for each pair of elements X and Y of S we define the distance $\rho(X, Y)$ between X and Y as the upper limit of the distances $\rho(x, y)$, where x and y are points of X and Y respectively, it readily follows that the space S' so obtained is compact, separable, metric and connected; and since it readily follows that every two "points" of S' are separated in S' by some third "point" of S', therefore† S' is an acyclic continuous curve.

6. EXISTENCE THEOREM. *If the space M is connected im kleinen, there exists an uncountable non-separated collection of cuttings of M.*

Let a and b be any two points of M, and for each positive number r which is less than the distance from a to b, let $S(a, r)$ denote the set of all points of M whose distance from a is equal to r and let $I(a, r)$ denote the set of all points at a distance $<r$ from a. Then for each r, $S(a, r)$ separates a and b in M. Let $R(a, r)$ denote the component of $M-S(a, r)$ containing a, let $R(b, r)$ denote the component of $M-\overline{R(a, r)}$ containing b, and let X_r denote the point set $\overline{R(a, r)}\cdot\overline{R(b, r)}$. Then clearly X_r separates a and b in M and we have

(i) $$X_r \subset F[R(a, r)] \subset S(a, r), \text{ and } X_r = F[R(b, r)];$$

(ii) $$R(a, r) \subset I(a, r).$$

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* See R. L. Moore, these Transactions, vol. 27 (1925), pp. 416–428.

† See R. L. Moore, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–307.

Obviously the collection of cuttings $[X_r]$ is uncountable. It remains to show that it is non-separated. Let X_{r_1} and X_{r_2} be any two elements of this collection and suppose $r_1 < r_2$. By (ii) it follows that $\overline{R(a, r_1)} \subset R(a, r_2)$. Thus $X_{r_1} \subset R(a, r_2)$, and therefore X_{r_2} does not separate X_r in M. From the inclusion $\overline{R(a, r_1)} \subset R(a, r_2)$ and (i) it follows that $\overline{R(b, r_2)} = R(b, r_2) + X_{r_2} \subset R(b, r_1)$, and consequently X_r does not separate X_{r_2} in M. Thus the collection $[X_r]$ is non-separated, and the theorem is proved.

Now since a may be any point whatever of M and since every neighborhood of a contains uncountably many of the sets $[X_r]$, it follows by §4, (δ), that every such neighborhood contains at least one set X_r which is an irreducible cutting of M. Thus we have the following

COROLLARY. *Every open subset of a connected and connected im kleinen point set M lying in a separable metric space contains an irreducible cutting I of M.*

This corollary answers a question raised by the author.*

As a result of this existence theorem it follows that the decomposition treated in §5 is always realisable in case M is locally connected; and in case M is a continuous curve, M may be decomposed upper semi-continuously into a collection S of the type attained in §5, and the decomposition space S' is a non-degenerate acyclic continuous curve.

7. **Concluding remarks.** Although it is easily seen with the aid of a very simple example that two cuttings X and Y of M may have the property that neither of them separates the other in M and yet the set $M_2(X)+X$ not be connected, where $M_2(X) \supset Y$, nevertheless the following lemma is true.

LEMMA. *If a and b are two points of M and $X_1, X_2, X_3, \cdots$ is any infinite sequence of distinct mutually exclusive sets each of which separates a and b in M and no one of which separates any other one, and we have*

$$X_1 < X_2 < X_3 < \cdots,$$

then the set of points $\sum_1^\infty M_1(X_i)$ is connected.

For if on the contrary this set of points is the sum of two mutually separated sets N_1 and N_2, then since a belongs to all of the sets $M_1(X_i)$, there exists an integer n such that $N_1 \cdot M_1(X_n) \neq 0 \neq N_2 \cdot M_1(X_n)$. Since by (2.2), (b), it follows that $M_1(X_n) \subset M_1(X_{n+1})$, therefore $N_1 \cdot M_1(X_{n+1}) \neq 0 \neq N_2 \cdot M_1(X_{n+1})$. Since these two sets are mutually separated, one of them, say $N_1 \cdot M_1(X_{n+1})$, contains X_n. But then it is easily seen that the sets $N_2 \cdot M_1(X_n)$ and $M - N_2 \cdot M_1(X_n)$ are mutually separated, contrary to the fact that M is connected.

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* See Fundamenta Mathematicae, vol. 13 (1929), p. 50, where the question is raised for *continuous curves M*. A solution of this problem for the case where M is a plane continuous curve has been given by J. H. Roberts; see these Transactions, vol. 32 (1930), p. 19.

With the aid of this lemma it can be shown without great difficulty that if X is any element of a non-separated collection G of subsets of M which is saturated in M relative to the property ρ defined by the collection G, then

(1) each component of $M - X$ is open in M;

(2) the components of $M - X$ are countable;

(3) X is a potentially regular element of G in M relative to G, i.e., a monotone decreasing sequence of neighborhoods $[U_i]$ of X exists such that $F(U_i)$ is a subset of a finite number of the elements of G and $X = \prod_1^\infty \overline{U}_i$;

(4) the potential order of X in M relative to G is equal to the number of components of $M - X$ when this number is finite, and is equal to ω (i.e., X is of increasing order) when and only when this number is infinite.

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CUT POINTS OF CONNECTED SETS AND OF CONTINUA[*]

BY

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1. **Introduction.** In this paper it will be shown, among other things, that the set G of all the cut points of any continuum M in a locally compact, metric, and separable space is a $G_{\delta\sigma}$, i.e., the sum of a countable number of G_δ sets.[†] From this result, with the aid of a well known theorem of W. H. Young's, it follows that if the set G is uncountable it must contain a perfect set.[‡] Thus the set of all cut points of any continuum is either vacuous, finite, countable, or of the power of the continuum.

The customary notation and terminology of point set theory will be employed. For example, $\overline{X}=X+X'$, where X' is the set of all limit points of the set X; $K\subset H$ means that K is a subset of H or that H contains K; $K\cdot H$ denotes the set of points common to K and H; $\rho(X, Y)$ denotes the distance from X to Y when X and Y are points, and denotes the minimum distance between X and Y when X and Y or either contains more than one point, i.e. the greatest lower bound of the aggregate of numbers $[\rho(x,y)]$, where x and y are points of X and Y respectively; and $\delta(M)$ denotes the diameter of the set M, i.e., the least upper bound of the aggregate $[\rho(x,y)]$, where x and y are two points of M. In addition, if X is a cutting of a connected set M, unless otherwise stated, the equation $M-X=M_a(X)+M_b(X)$ is to be interpreted as meaning that $M-X$ is the sum of the two mutually separated sets $M_a(X)$ and $M_b(X)$ containing the sets A and B respectively. When this equation is true, X is said to separate A and B in M.

All point sets considered are assumed to lie in a locally compact, metric, and separable space.

2. **Cut points and regular points.** We prove the following theorem:

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[*] Presented to the Society, December 27, 1928. Received by the editor of the Bulletin in December, 1928, accepted for publication in the Bulletin, and subsequently transferred to the Transactions.

[†] It has been shown by C. Zarankiewicz that the set of all cut points of any *continuous curve* is an F_σ but that this is not true of continua in general; see *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), pp. 124–171; see pp. 163, 164.

[‡] According to results due to F. Bernstein (Leipziger Berichte, vol. 60 (1908), p. 325), there exist uncountable sets, even uncountable linear sets, which contain no perfect set. The theorems in this paper show, however, that such a set cannot be the set of cut points of any continuum.

THEOREM 1. *If A and B are any two mutually exclusive closed subsets of a connected set M and K is the set of all those points of M which separate A and B in M, then $K+A+B$ contains every point P of $\overline{K}$ which is a regular point*[*] *of $M+P$.*

Let P be any limit point of K which is not in $A+B$ and which is a regular point of $M+P$. As shown in my paper *Concerning connected and regular point sets*[†] either there exists a sequence $X_1, X_2, X_3, \cdots$ of points of K having P as its sequential limit point and such that, for every positive integer i,

$$M - X_i = M_a(X_i) + M_b(X_i), \quad \sum_{n=1}^{i-1} X_n \subset M_a(X_i), \text{ and } \sum_{n=i+1}^{\infty} X_n \subset M_b(X_i),$$

or such a sequence exists such that, for each i,

$$\sum_{n=1}^{i-1} X_n \subset M_b(X_i) \text{ and } \sum_{n=i+1}^{\infty} X_n \subset M_a(X_i).$$

The two cases are alike so we shall consider only the former. Let $E = \sum M_a(X_i)$, and let $F = M - E$. Then $F = \Pi\, M_b(X_i)$, i.e. F is identically the set of points common to all the sets $[M_b(X_i)]$. Thus no point of E is a limit point of F, because each point of E belongs to a set $M_a(X_k)$, and $F \subset M_b(X_k)$ for every k. I shall now show that no point of F except possibly P (in case P belongs to M) is a limit point of E. Suppose, on the contrary, that such a limit point Q of E does exist. Let ϵ be a positive number which is $< \rho(P, Q+A+B)$. Since P is a regular point of $M+P$, there exists a finite subset U of M which ϵ-separates P in $M+P$, i.e., $(M+P)-U = M_p + M_0$, where M_p and M_0 are separated, $M_p \supset P$, and $\delta(M_p) < \epsilon$. Now[‡] $M_0 + U$ is the sum of a finite number of mutually separated connected point sets $E_1, E_2, E_3, \cdots, E_n$. One of these, say E_k, contains Q, and Q is not a limit point of $M-E_k$. But there exists an integer m such that for every integer $j>m$, $X_j \subset M_p$. And since $Q \subset M_b(X_j)$, $E_k \supset Q$, and $E_k \cdot X_j=0$, then $E_k \subset M_b(X_j)$ for every $j>m$. Hence Q is not a limit point of $\sum_{j=m+1}^{\infty} M_a(X_j)$, because this set of points is a subset of $M-E_k$. But clearly Q is not a limit point of $\sum_{j=1}^{m} M_a(X_j)$, because $Q \subset F \subset M_b(X_j)$ for every j. Therefore Q is not a limit point of E, contrary to supposition. Thus no point of F except possibly P can be a limit point of E.

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[*] The point P of a connected point set M is called a Menger regular point of M, or simply a regular point of M, if for each $\epsilon > 0$, P can be ϵ-separated in M by some finite subset U of M, i.e., a finite subset U of M exists such that $M - U = M_p(U) + M(U)$, where $M_p(U)$ and $M(U)$ are mutually separated, $M_p(U) \supset P$, and $\delta[M_p(U)] < \epsilon$. If the ϵ-separating set U can, for every $\epsilon > 0$, be chosen of power $\leqq n$ but cannot, for every ϵ, be chosen of power $< n$, then P is a point of order n of M. See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306, and P. Urysohn, *Sur la ramification des lignes Cantoriennes*, Comptes Rendus, vol. 175 (1922), p. 481.

[†] Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 685–689, see proof of Theorem 1. The proof here given is somewhat similar to the one just referred to. R. L. Wilder has recently published some interesting extensions of Theorem 1 in my paper here cited; see R. L. Wilder, *On connected and regular point sets*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 649–655.

[‡] See Knaster and Kuratowski, *Remark on a theorem of R. L. Moore*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 647–649; see also an abstract of mine in the Bulletin of the American Mathematical Society, vol. 33 (1927), p. 388.

Now P must belong to F, for if not, then E and F are mutually separated; and since $E+F=M$, this contradicts the fact that M is connected. Hence $P \subset F$; and since E and $F-P$ are mutually separated and contain A and B respectively, and $E+(F-P)=M-P$, therefore P separates A and B in M and hence belongs to K.

Corollary 1a. *If M is a continuum, then every point of $\bar{K}$ which is not in $K+A+B$ is a non-regular point of M.*

Theorem 1 does not hold true, even in case M is a continuum, when we substitute the words "point of connectivity im kleinen" for the words "regular point." For let I be the interval $(0,2)$ of the X-axis; let A and B be the end points of I; let L be the straight line interval from $(1,0)$ to $(1,1)$; let P be the point $(1,0)$; for each positive or negative integer n, let L_n be the straight line interval from $(1+1/(2n),0)$ to $(1+1/(2n),1)$; and let M be the continuum $I+L+\sum L_n$. Then the set K of points of M which separate A and B in M is identical with the set $I-(A+B+P)$; and P is a limit point of K and is a point of connectivity im kleinen of M; but P does not belong to K.

Theorem 2. *Let N be any closed subset of a connected point set M, and let K be the set of all points of M which separate*[*] *N in M. Then $K+N$ contains every point P of $\bar{K}$ which is a regular point of $M+P$.*

Let P be any point of $\bar{K}$ which is not in N and which is a regular point of $M+P$, let ϵ be a positive number which is $<\rho(P,N)$, and let U be a finite subset of M which ϵ-separates P in $M+P$. Then $M-U=M_p+M_n$, where M_p and M_n are separated and contain P and N respectively. Now M_p contains an infinite sequence $X_1, X_2, X_3, \cdots$ of points of K having P for its sequential limit point. For each i, $M-X_i=M_1(X_i)+M_2(X_i)$, where $M_1(X_i)$ and $M_2(X_i)$ are separated and $N \cdot M_1(X_i) \neq 0 \neq N \cdot M_2(X_i)$. Now since[†] M_n+U is the sum of a finite number of connected point sets, it follows that there exist two point sets K_a and K_b and an infinite subsequence

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[*] The point P of a connected set M is said to separate a given subset N of M in M if $M-P$ is separated between some two points of N, i.e., $M-P$ is the sum of two mutually separated sets M_1 and M_2, where $N \cdot M_1 \neq 0 \neq N \cdot M_2$; see R. L. Wilder, loc. cit.

[†] See the reference to Knaster and Kuratowski above.

$X_{n_1}, X_{n_2}, \cdots$ of the sequence $[X_n]$ such that $K_a+K_b=M_n+U$ and such that, for every i, $K_a \subset M_1(X_{n_i})$ and $K_b \subset M_2(X_{n_i})$. Hence if $A=N \cdot K_a$ and $B=N \cdot K_b$, then A and B are mutually exclusive, closed, and nonvacuous subsets of M, and every point of $\sum X_{n_i}$ separates A and B in M. Then since P is a limit point of $\sum X_{n_i}$ and is a regular point of $M+P$, it follows by Theorem 1 that P belongs to M and separates A and B in M. Therefore P separates N in M and hence belongs to K. This completes the proof.

THEOREM 3. *Let A and B be any two mutually exclusive closed subsets of a continuum M, and let K be the set of all those points of M which separate A and B in M. Then $K+A+B$ contains every point P of $\overline{K}$ having the property that every subcontinuum of M containing P contains at least two points at which M is connected im kleinen.*

Let P be a point of $\overline{K}$ not in $A+B$ which has the property mentioned in this theorem. Let the points and point sets $X_1, X_2, X_3, \cdots, E$, and F be selected and defined exactly as in the proof of Theorem 1. Then, just as in that proof, no point of E is a limit point of F. Likewise no point of $F-P$ can be limit point of E. For suppose some point Q of $F-P$ is a limit point of E. Let R be a compact neighborhood of P such that $\overline{R}$ does not contain Q. Then since for no i can Q be a limit point of $M_a(X_i)$, it readily follows that if, for each i, $I(X_i,X_{i+1})$ denotes the set of points $M_b(X_i) \cdot M_a(X_{i+1})+X_i+X_{i+1}$, then both Q and P belong to the sequential limiting set L of the sequence of continua* $[I(X_i,X_{i+1})]$. Now with the aid of a theorem of Janiszewski's† and a theorem of Lubben's,‡ it follows that $L \cdot \overline{R}$ contains a continuum H containing P and at least one point of the boundary of R. Since for each i, $L \cdot M_a(X_i)=0$, clearly $H \subset F$. Now by hypothesis, H contains a point C, distinct from P, at which M is connected im kleinen. Since C belongs to L, it follows that there exists an integer k and a point D of $I(X_k,X_{k+1})$ which can be joined to C by a subcontinuum I of M which does not contain P. There exists an integer $j>k$ such that X_j does not belong to I. But then C belongs to $M_b(X_j)$ (for $C \subset F \subset M_b(X_i)$, for every i), D belongs to $M_a(X_j)$, and I is a connected subset of $M_a(X_j)+M_b(X_j)$ containing both C and D. Clearly this is impossible, because $M_a(X_j)$ and $M_b(X_j)$

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* See my paper *Concerning the cut points of continua*, these Transactions, vol. 30 (1928), pp. 597–609, Theorem 1.

† Z. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), vol. 16 (1912), p. 109.

‡ R. G. Lubben, *Concerning limiting sets in abstract spaces*, these Transactions, vol. 30 (1928), pp. 668–685; see Theorems 11 and 12.

are mutually separated. Thus the supposition that $F-P$ contains a limit point of E leads to a contradiction. Hence E and $F-P$ are mutually separated; and since $E \supset A$, $F-P \supset B$, and $E+(F-P)=M-P$, it follows that P separates A and B in M and hence belongs to K.

3. **Cardinal number and Borel classification of the set of cut points.** We shall prove the following theorem:

THEOREM 4. *Let A and B be any two points, or, indeed, any two closed mutually exclusive subsets, of a continuum M, and let K be the set of all points of M which separate A and B in M. Then K is the sum of a G_δ set*[*] *and a countable set.*

Let E be the set of all those points of M which can, for each $\epsilon>0$, be ϵ-separated in M by some two points of K, i.e., the set of all points of M of order $\leqq 2$ relative to K. It follows readily with the aid of a theorem of Menger's[†] that if E exists it is a G_δ set. And since $A+B$ is closed, $H=E-E\cdot(A+B)$ is a G_δ set. Now since every point of E is a limit point of K and is a regular point of M, then by Theorem 1, $K+A+B\supset E$. Hence $H\subset K$. And by a theorem of the author's,[‡] $K-H$ is countable. This completes the proof.

COROLLARY[§] 4a. *If K is uncountable, it contains a perfect set.*

THEOREM 5. *The set G of all the cut points of any continuum M is a $G_{\delta\sigma}$, i.e., the sum of a countable number of G_δ sets.*

Since our space is metric and separable, M itself is[||] separable and hence contains a countable set D such that $M=\bar{D}$. Let H be the collection of all possible pairs of points of D, and for each pair (A,B) of points in H, let K_{ab} be the set of all points of M which separate A and B in M. By Theorem 4, $K_{ab}=$ a G_δ set $+$ a countable set, for every pair (A,B) in H. Then since H is countable, it follows that $\sum_H K_{ab}$ is a $G_{\delta\sigma}$. Now if X is any point of G, $M-X=M_1+M_2$, where M_1 and M_2 are mutually separated. And if P_1 and P_2 are points of M_1 and M_2, respectively, then since $P_1\cdot M_2' =0=P_2\cdot M_1'$, and $P_1+P_2 \subset \overline{D}$, then D is a subset of neither M_1+X nor M_2+X. Hence there exist points A and B of D in M_1 and M_2 respectively. Therefore X separates A and B in M and hence belongs to K_{ab} and to $\sum K_{ab}$. Thus $G \equiv \sum K_{ab}$, and hence G is a $G_{\delta\sigma}$.

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[*] A G_δ set is a point set which is the common part of some family of open sets.

[†] See K. Menger, loc. cit., Theorem 3.

[‡] See *Concerning the cut points of continua*, loc. cit., Theorem 7; see also my paper *Concerning collections of cuttings of connected point sets*, Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 87–104, Theorem 12.

[§] This corollary follows from Theorem 4 and Young's theorem that every G_δ set containing a subset which is dense in itself contains a perfect set; see W. H. Young, Leipziger Berichte, vol. 55 (1903), p. 287.

[||] See W. Gross, *Zur Theorie der Mengen, in denen ein Distanzbegriff definiert ist,* Wiener Sitzungsberichte, vol. 123 (1924), pp. 801–819.

Corollary* 5a. *If G is uncountable, it contains a perfect set.*

We note here the following fact which, in a certain sense, is a converse proposition to Theorem 4:

If K is any set which lies within the linear interval $I(A, B)$ and is the sum of a G_δ set and a countable set, then there exists a continuum M such that K is identically the set of all those points of M separating A and B in M.

We note also the fact that the set of all the cut points of a continuum is not necessarily the sum of a G_δ set and a countable set. Indeed, *the set of all cut points of the universal acyclic continuous curve of Wazewski is not the sum of a G_δ set and a countable set.*

4. **Concluding remarks.** Theorem 5 and Corollary 5a are propositions concerning the set G of all the cut points of a continuum M. We may obtain the same conclusions for the set H of all the *local separating points*† of a continuum M. We state the following propositions concerning the local separating points of continua.

(i) Every continuum M contains a countable collection $[M_i]$ $(i=1,2,3, \cdots)$ of compact subcontinua such that if H is the set of all local separating points of M and, for each i, G_i is the set of all cut points of M_i, then (1) $\sum_1^\infty G_i \subset H$, (2) $H-\sum G_i$ is countable, and (3) for each i, $M_i \cdot (M-M_i)$ contains exactly two points.

(ii) The set H of all the local separating points of any continuum M is a $G_{\delta\sigma}$; and if H is uncountable, it contains a perfect set.

(iii) Every continuous curve M contains a countable collection $[M_i]$, $(i=1,2,3, \cdots)$ of compact continuous curves such that if H is the set of all local separating points of M and, for each i, G_i is the set of all cut points of M_i, then $\sum_1^\infty G_i \equiv H$.

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* This corollary can be deduced from Theorem 5 using Young's theorem (loc. cit.), or it follows also from Corollary 4a and the author's theorem (see *Concerning the cut points of continua*, loc. cit., Theorem 3) that if G is any uncountable set of cut points of M, then some two points A and B of G are separated in M by uncountably many points of G.

† The point P of a continuum M is a local separating point of M if there exists a compact neighborhood R of P such that $M \cdot \overline{R} - P$ is separated between some two points of the component of $M \cdot \overline{R}$ which contains P; see my paper *Local separating points of continua*, in Monatshefte für Mathematik und Physik, vol. 35, No. 2.

Result (i) is readily deduced from the theorems in my paper *Local separating points of continua* (loc. cit.)—in particular from the result that all save possibly a countable number of the local separating points of any continuum M are points of order two of M. Result (ii) follows immediately from (i) and Theorem 5 and Corollary 5a above. Result (iii) was established incidentally by the author in proving the theorem* that the set H of all the local separating points of any continuous curve is an F_σ.

It has been shown by the author† that all save possibly a countable number of the cut points of any *continuum* M are points of order two of M. That this theorem does not hold true for *connected sets* M, even in case M is irreducibly connected between some two of its points, is shown in the example given by Vietoris‡ of a totally discontinuous function $y=\omega(x)$ which has a connected graph. That this theorem does not hold true even for *connected and connected im kleinen sets* M is shown by the following example. Let K be a non-dense perfect set on the interval $I(0,1)$ of the X-axis not containing the end points A and B of I. Let $S_1, S_2, S_3, \cdots$ be the complementary segments of K in I. For each i, let E_i be an ellipse with minor axis $\overline{S}_i$ and with major axis two units in length. Let $M=I+\sum_1^\infty E_i$. Then M is connected and connected im kleinen, and every point of K separates A and B in M; but clearly no point of K is a regular point of M.

Recently I have made a study of the structure of connected and connected im kleinen point sets and, among other results, have found the following theorem:

In a connected and connected im kleinen point set M, every connected set of cut points is arcwise connected.§

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* See my paper *Concerning points of continuous curves defined by certain im kleinen properties* Mathematische Annalen, vol. 102 (1929), pp. 133–336.

† See *Concerning the cut points of continua*, loc. cit., Theorem 7.

‡ L. Vietoris, *Stetige Mengen*, Monatshefte für Mathematik und Physik, vol. 31 (1921), p. 202; see also Knaster and Kuratowski, *Sur quelques propriétés topologiques des fonctions dérivées*, Rendiconti del Circolo Matematico di Palermo, vol. 49 (1925).

§ For special cases of this theorem, see an abstract by C. M. Cleveland in the Bulletin of the American Mathematical Society, vol. 32 (1926), p. 420, where the theorem is stated for *plane continuous curves M*, and my paper *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194, Theorem 8, where the theorem, in more general form, is proved for continuous curves M in n-dimensional space.

It is of interest in this connection to note the theorem, easily deducible from the work of F. Frankl*, that *Every connected point set M each point of which is of order $\leqq 2$ of M is arcwise connected.*

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* See *Ueber die zusammenhängenden Mengen von höchstens zweiter Ordnung*, Fundamenta Mathematicae, vol. 11 (1927), pp. 96–104.

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ON THE REGULAR POINTS OF A CONTINUUM

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Reprinted from the
Transactions of the American Mathematical Society
Vol. 33, No. 1, pp. 252-262

ON THE REGULAR POINTS OF A CONTINUUM[*]

BY

W. L. AYRES

I. INTRODUCTION

1. We consider a compact, connected metric space M which we shall call the continuum M. A point p of M is said to be a *regular point* if for each $\epsilon>0$ there exists a neighborhood U_p of p (i.e., an open subset of M containing p) such that $d(U_p)<\epsilon$ and $F(U_p)$ consists of a finite number of points. The point p is said to be a *point of order* α if (1) for each $\epsilon>0$ there exists a neighborhood U_p such that $d(U_p)<\epsilon$ and the cardinal number of $F(U_p)\leqq\alpha$, and (2) α is the smallest cardinal number for which (1) is true. Regular points of no finite order are called points of order ω. Let M^α denote the set of all points of M of order α. Then

$$M = M^1 + M^2 + \cdots + M^\omega + M^{\aleph_0} + M^c.$$

These definitions were introduced by Urysohn and Menger[†] several years ago and since that time have been studied in a number of papers.

One of the interesting studies in this theory is that of the distribution and structure of the various sets M^α. It is of course quite obvious that M may be composed entirely of points of order 2 or entirely of points of order c. Urysohn[‡] has given examples to show that M may be composed entirely of points of order ω or entirely of points of order $\aleph_0$. Except for these four orders this is not possible, a consequence of a theorem proved independently by Urysohn,[§] G. T. Whyburn[||] and H. Künneth[¶] that if all points of M are of order $\leqq n$, then the points of order $\leqq \frac{1}{2}n+1$ are dense in M.

Since $M\neq M^n (n\neq 2)$, it would be interesting to know more of the distribution of the points of M^n. Whyburn[**] has shown that M^n is punctiform and has raised the question as to whether it is of dimension zero. In the present paper we will answer this question in the affirmative as a special case of our theorem that $M^n+M^{n+1}+\cdots+M^{2n-3}$ is a set of dimension zero. As corollaries of this theorem we obtain most of the previously known results concerning the distribution of the points of finite orders. In the last section of the paper we complete our study of the structure of the sets of various finite orders by examining the set M^2. Here we prove that M^2 is composed of a set of dimension zero plus a countable set of arcs.

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[*] Presented to the Society, September 9, 1930; received by the editors September 9, 1930.

[†] P. Urysohn, Comptes Rendus, vol. 175 (1922), pp. 481–483; and K. Menger, Monatshefte für Mathematik und Physik, vol. 33 (1923), pp. 148–160.

[‡] *Mémoire sur les multiplicités cantoriennes*, 2ème Partie, Verhandelingen, Akademie van Wetenschappen, Amsterdam, vol. 13 (1927), No. 4, pp. 109–115.

[§] Ibid., pp. 105–9.

[||] *On regular points of continua* etc., Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 218–224.

[¶] *Ein Theorem der Kurventheorie*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 149–152.

[**] Loc. cit.

2. **Notation.** Capitals will denote sets, lower case letters denote individual elements which are either points or numbers. The usual notation of the theory of sets will be employed. Below we will list some special notation which, while not new, is not universally employed by writers in this field and thus needs definition.

$p \mathcal{E} N \equiv p$ is an element of the set N.
p non-$\mathcal{E} N \equiv p$ is not an element of the set N.
$\rho(p, q) \equiv$ distance between the points p and q.
$\rho(M, N) \equiv$ greatest lower bound of $\rho(p, q)$ for $p \mathcal{E} M$ and $q \mathcal{E} N$.
$d(M) \equiv$ diameter of $M \equiv$ least upper bound of $\rho(p, q)$ for $p+q \subset M$.
$S(p, \eta) \equiv$ set of all points q such that $\rho(p, q) < \eta$.
$\mathcal{C}(p, N) \equiv$ component of set N containing the point p.
$F(N) \equiv \overline{N} \cdot (M-N) + \overline{M-N} \cdot N \equiv$ frontier or boundary of N.
$C(N) \equiv M-N \equiv$ complement of N.
$\dim_p E$, $\text{order}_p E \equiv$ dimension of set E at p, order of E at p.

II. The structure of M^n, $n \neq 2$

3. Theorem. *For any integer $n>2$, the set of all points p of a continuum M such that $n \leqq \text{order}_p M \leqq 2n-3$ is a zero-dimensional set (or vacuous).**

Let E denote the set of points p such that $n \leqq \text{order}_p M \leqq 2n-3$. Given $\epsilon>0$ and $p \mathcal{E} E$, we shall show the existence of a neighborhood $U_p \subset S(p, \epsilon)$ such that $F(U_p) \cdot E = 0$, i.e., $\dim E = 0$. If p is a point of order m, by an *order neighborhood* of p we shall mean a neighborhood of p whose boundary consists of exactly m points.

Let $Z_1 \subset S(p, \frac{1}{2}\epsilon)$ be an order neighborhood of p. If $E \cdot F(Z_1) = 0$, our proof is complete. If not, let $E \cdot F(Z_1) = q_1 + q_2 + \cdots + q_{s_1}$ $(s_1 \leqq 2n-3)$. Let U_k be an order neighborhood of $q_k (1 \leqq k \leqq s_1)$ such that $\overline{U}_k \subset S(q_k, r_k)$, where r_k is the smaller of the numbers $\epsilon/4$ and $\frac{1}{2}\rho(q_k, F(Z_1)+p-q_k)$. We see that $\overline{U}_i \cdot \overline{U}_j = 0$ if $i \neq j$. Let $V_k = Z_1 \cdot U_k$ and $W_k = C(\overline{Z}_1) \cdot U_k$. Since $q_k \mathcal{E} E$, $F(U_k)$

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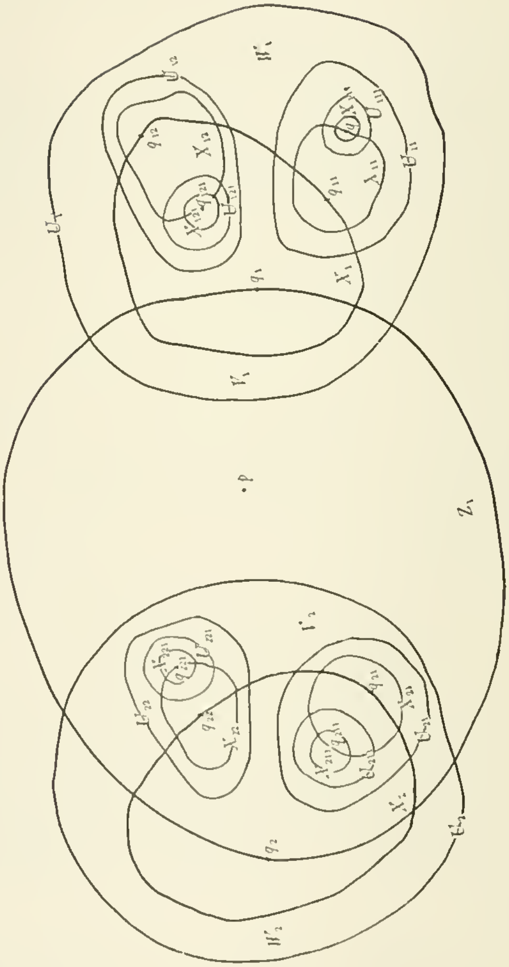
* Professor K. Menger has called attention to the fact that no use is made of the condition that M be connected and compact in the proof of this theorem. Hence the theorem is true for *any* metric space M.


W_1
U_{12}
q_{12}
X_{12}
U_1
X_{121}
q_{121}
U_{121}
X_{111}
U_{111}
U_{11}
X_{11}
q_{11}
X_1
q_1
V_1
p
Z_1
V_2
X_{221}
q_{221}
U_{221}
U_{22}
q_{22}
X_{22}
q_{21}
X_{21}
U_{21}
X_{211}
q_{211}
U_{211}
X_2
q_2
U_2
W_2

consists of $\leqq 2n-3$ points. And since $F(V_k)+F(W_k)=F(U_k)+q_k$ and $F(V_k)\cdot F(W_k)=q_k$, either $F(V_k)$ or $F(W_k)$ consists of $\leqq n-1$ points. Let I_1 be the set of all integers i such that $F(W_i)$ consists of $\leqq n-1$ points, and let J_1 be the set of all integers $1\leqq j\leqq s_1$ not in I_1. Let X_k be an order neighborhood of q_k such that $\overline{X}_k\subset U_k$. Now let

$$Z_2 = Z_1 + \sum X_i - \sum \overline{X}_j,$$

where the first summation extends over I_1 and the second over J_1.

If $F(Z_2)\cdot E=0$, then Z_2 is the desired neighborhood U_p. If not, we have $F(Z_2)\cdot E\subset\sum_k F(X_k)$. Let $F(X_k)\cdot F(Z_2)\cdot E=q_{k1}+q_{k2}+\cdots+q_{ks_2k}$. For each point $q_{im}(i\in I_1)$, let U_{im} be an order neighborhood of q_{im} such that $\overline{U}_{im}\subset W_i\cdot S(q_{im}, r_{im})$, where r_{im} is the smaller of the numbers $\epsilon/8$ and $\frac{1}{2}\rho(q_{im}, F(X_i)-q_{im})$. For each point $q_{jm}(j\in J_1)$, let U_{jm} be an order neighborhood of q_{jm} such that $\overline{U}_{jm}\subset V_{jm}\cdot S(q_{jm}, r_{jm})$, where r_{jm} is the smaller of the numbers $\epsilon/8$ and $\frac{1}{2}\rho(q_{jm}, F(X_j)-q_{jm})$. We see that $\overline{U}_{k_1m_1}\cdot\overline{U}_{k_2m_2}=0$ unless $k_1=k_2$, $m_1=m_2$. Let $V_{km}=Z_2\cdot U_{km}$ and $W_{km}=C(\overline{Z}_2)\cdot U_{km}$. Since $q_{km}\in E$, $F(U_{km})$ consists of $\leqq 2n-3$ points. Then either $F(W_{km})$ or $F(V_{km})$ consists of $\leqq n-1$ points. Let I_2 be the set of all pairs (k, m) such that $F(W_{km})$ consists of $\leqq n-1$ points. Let J_2 be the set of all pairs (k, m) for which U_{km} is defined that are not in I_2. Let X_{km} be an order neighborhood of q_{km} such that $\overline{X}_{km}\subset U_{km}$. Now let

$$Z_3 = Z_2 + \sum X_{km} - \sum \overline{X}_{km},$$

where in the first summation $(k, m)\in I_2$ and in the second $(k, m)\in J_2$.

If $F(Z_3)\cdot E=0$, Z_3 is the desired neighborhood U_p. If not, we repeat this process on the points of $F(Z_3)\cdot E=F(Z_3)\cdot E\cdot\sum_k\sum_m F(X_{km})$. Continuing this process, at some stage we reach a neighborhood Z_i such that $F(Z_i)\cdot E=0$ or the process continues indefinitely.

In case the process continues indefinitely, we define a monotonic increasing sequence of neighborhoods of p as follows:

$$Y_1 = Z_1 - \sum_{k=1}^{s_1}\overline{U}_k,\quad Y_2 = Z_2 - \sum_{k=1}^{s_1}\sum_{m=1}^{s_{2k}}\overline{U}_{km},$$

and similarly we define Y_t for each positive integer t. Now let

$$U_p = \sum_{t=1}^{\infty} Y_t.$$

Since $U_p\subset Z_1+\sum_k U_k\subset S(p, \epsilon)$, then U_p is the desired neighborhood if $F(U_p)\cdot E=0$. Every point of $F(Z_t)$ not belonging to E is a point of $F(Y_t)$ and of every $F(Y_s)$ for $s\geqq t$. Then the points of $F(U_p)$ are of two classes:

(a) points which belong to $F(Y_s)$ for every s greater than some fixed integer, and (b) $\Pi_{t=1}^{\infty}(F(U_p)-F(Y_t))$. Just above it was seen that no point of the class (a) is a point of E. Now as

$$F(U_p) - F(Y_t) \subset \sum U_{k_1k_2\cdots k_t},$$

we have that every point of $F(U_p)$ of class (b) belongs to the set

$$H = \prod_{t=1}^{\infty} \sum U_{k_1k_2\cdots k_t}.$$

Now consider any point y of H. Each neighborhood U_{km} is a subset of V_k or W_k according as $k \mathcal{E} I_1$ or J_1, i.e. U_{km} is a subset of a neighborhood of the first stage whose boundary contains $\leqq n-1$ points. Similarly for the neighborhoods $U_{k_1k_2\cdots k_t}$ of any stage. Then at each stage y belongs to a neighborhood whose boundary contains $\leqq n-1$ points. And as the diameters of the neighborhoods approach zero, it follows that $\text{order}_y M \leqq n-1$. Hence y non-$\mathcal{E} E$. Then U_p is the desired neighborhood of p as

$$U_p \subset S(p, \epsilon) \text{ and } F(U_p) \cdot E = 0.$$

4. This section proves corollaries to the preceding theorem.

Corollary 1. *For each positive integer $n \neq 2$, the set of all points of M of order n is zero-dimensional (or vacuous).*

Since a subset of a vacuous or zero-dimensional set is necessarily of the same type, for $n>2$ this follows from our theorem. That the set M^1 is of this type has been shown by Menger and Urysohn.*

Corollary 2.† *There exists no continuum all of whose points are of order $n \neq 2$.*

If M is a continuum, $\dim M \geqq 1$. Hence $M - M^n \neq 0$ for any $n \neq 2$.

Corollary 3. *The simple closed curve is the only (compact) continuum all of whose points are of the same finite order.*

Corollary 4. *If the order of every point of the continuum M is $\leqq m$, then the points of order $\leqq \frac{1}{2}m+1$ are dense in M.*

From our theorem it follows that the set of all points p such that $\frac{1}{2}m+1 < \text{order}_p M \leqq m$ is zero-dimensional. Hence it contains no open subset and the remaining points are dense in M.

5. **Remarks.** It may be noticed in the proof of the preceding theorem

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* K. Menger, Mathematische Annalen, vol. 95 (1925), p. 293; and P. Urysohn, Second reference, p. 79.

† The corollaries 2, 3 and 4 are all known results. See the papers cited in the Introduction.

that, while at each stage the neighborhoods have only finite boundaries, the final neighborhood U_p may have an infinite boundary. This raises the question as to whether it is possible to select U_p with only a finite boundary. This is not always true. If we join two of the Sierpinski triangle curves* at the vertices we have a continuum containing only points of orders 3 and 4. Now if we take $n=4$ the only neighborhoods U_p such that $F(U_p)\cdot E=0$ have infinite boundaries.

We have seen that $\dim M^n=0$ for $n>2$. It would be interesting now to determine the conditions under which $\dim \sum_{n>2} M^n=0$.

Urysohn has constructed very interesting examples of continua containing points of orders n and $2n-2$ only for any $n>2$. This should lead to a study of continua containing points of orders m and n only $(m>n>2)$. From our theorem or Corollary 4, it follows that such a continuum can exist only if $m\geqq 2n-2$. It would be interesting to determine whether it can exist if $m\neq k(n-1)$. Also it seems likely that in such a continuum the points of order m are countable.

III. The structure of M^2

6. Lemma. *If p is a point of M^2 and a cut point im kleinen of M and* $\dim_p M^2>0$, *then M contains an arc one of whose end points is p such that the arc-segment*† *is an open subset of M.*

As p is a cut point im kleinen of M, there is a neighborhood Z_p such that $\mathcal{C}(p, \bar{Z}_p)=R+S$, where R and S are continua and $R\cdot S=p$. Then either $\dim_p R\cdot M^2>0$ or $\dim_p S\cdot M^2>0$ and we suppose the former. There exists a neighborhood U_p such that if the neighborhood $V_p\subset U_p$, then

$$F(V_p)\cdot M^2\cdot R\neq 0.$$

As $p\mathcal{E}M^2$ there is a neighborhood W_p such that $\bar{W}_p\subset U_p$ and $F(W_p)\cdot R$ is a single point r and $R-\bar{W}_p\neq 0$. Let $G=\bar{W}_p\cdot R$. As $r\mathcal{E}M^2$ and $R-\bar{W}_p\neq 0$, the point $r\mathcal{E}G^1$, i.e. the points of G of order 1 with respect to G. As $p\mathcal{E}G^1$ and $M^2\cdot G=G^2+p+r$, we will complete our proof by establishing the next lemma.

7. Lemma. *If G is a continuum and* (a) *p and r are end points of G, and* (b) *for any neighborhood U_p such that r non-$\mathcal{E}\bar{U}_p$, the set $F(U_p)\cdot G^2\neq 0$, then G is an arc with end points p and r.*

Let Q denote the set consisting of $p+r+$those points of G^2 which separate p and r in G. We shall prove that Q is a closed set. Let $[q_i]$ be a sequence of distinct points of Q such that $\lim q_i = q$. We shall show that $q \mathcal{E} Q$. This is true if $p=q$ or $r=q$. If $p \neq q \neq r$, there are two cases:

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* Comptes Rendus, vol. 160 (1915), pp. 302–5; and Prace Matematyczno-Fizyczne, vol. 27 (1915), pp. 77–86.

† By the *arc-segment* is meant the arc minus its end points.

Case I. There is an infinite subsequence $[q_{i_k}]$ such that

$$G - q_{i_k} = G_{pk} + G_{rk}, \quad G_{pk} \cdot \overline{G}_{rk} + \overline{G}_{pk} \cdot G_{rk} = 0,$$
$$p \mathcal{E} G_{pk}, \quad q + q_{i_{k+1}} + r \subset G_{rk}.$$

Since $q_{i_k} \mathcal{E} G^2$, G_{pk} and G_{rk} are connected and $G_{pk} \subset G_{pk+1}$. Let

$$X_p = \sum_{k=1}^{\infty} G_{pk}.$$

The set X_p is a neighborhood of p* and q non-$\mathcal{E} X_p$. Suppose $F(X_p) \supset t \mathcal{E} G^2 + r$, $t \neq q$. Let $[t_n]$ be a sequence such that $t_n \mathcal{E} G_{pn}$, $\lim t_n = t$. As $t \mathcal{E} \Pi G_{rk}$ it follows that if N is any subcontinuum of G containing t and t_n, then $q_{i_j} \mathcal{E} N$ for $j \geqq n$. Then N contains q. For this reason G is not locally connected at t. But $t \mathcal{E} G^2 + r$ and a continuum is locally connected at every point of finite order. Hence t cannot exist. By condition (b), $F(X_p) \cdot G^2 \neq 0$. Then $q \mathcal{E} G^2$.

Suppose there exists a neighborhood U_q of q such that both p and r belong to one component L of $G - U_q$. Then $L \supset \sum q_{i_k}$. But as $\lim q_{i_k} = q$, for k sufficiently large $q_{i_k} \subset U_q$. But this is absurd as $L \cdot U_q = 0$. Hence, for any neighborhood U_q such that $p + r \subset G - U_q$, p and r belong to different components of $G - U_q$. Now let $[U_n]$ be a set of neighborhoods of q such that $\overline{U}_{n+1} \subset U_n$, $d(U_n) < 1/n$, $F(U_n)$ consists of two points, $G - U_1 \supset p + r$. It is easily seen that

$$G - U_n = \mathcal{C}(p, G - U_n) + \mathcal{C}(r, G - U_n);$$

and thus

$$\mathcal{C}(p, G - U_{n+1}) \supset \mathcal{C}(p, G - U_n),$$
$$\mathcal{C}(r, G - U_{n+1}) \supset \mathcal{C}(r, G - U_n).$$

Then

$$G - q = \sum_n \mathcal{C}(p, G - U_n) + \sum_n \mathcal{C}(r, G - U_n).$$

Hence $q \mathcal{E} Q$.

Case II. There is an infinite subsequence $[q_{i_k}]$ such that

$$G - q_{i_k} = G_{pk} + G_{rk}, \quad \overline{G}_{pk} \cdot G_{rk} + G_{pk} \cdot \overline{G}_{rk} = 0,$$
$$p + q + q_{i_{k+1}} \subset G_{pk}, \quad r \mathcal{E} G_{rk}.$$

Let

$$P = \sum_{k=1}^{\infty} G_{rk}.$$

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* For this lemma we consider G as our space. Relative to G alone, the set X_p is open and hence is a neighborhood.

Exactly as in Case I we may show that if $F(P) \supset t\mathcal{E}G^2+p$, $t\neq q$, then G is not locally connected at t. We have thus

$$(1) \qquad F(P)\cdot(G^2 - q + p) = 0.$$

The set $G-\overline{P}$ is a neighborhood of p, and r non-$\mathcal{E}F(G-\overline{P})$. We have $F(G-\overline{P})\subset F(P)$. From (1) then,

$$F(G - \overline{P})\cdot(G^2 - q) = 0;$$

and, from condition (b), we must have $q\mathcal{E}G^2$. As in Case I we may show that q separates p and r in G.

We have shown that the set Q is closed. Let H be a subcontinuum of G irreducible between p and r. Suppose there exists a point $x\mathcal{E}H-Q$, and let $G_x=\mathcal{C}(x, H-Q)$. Let $y\mathcal{E}\overline{G}_x\cdot Q$. Since $p\mathcal{E}G^1$, there is a neighborhood Z_p such that $F(Z_p)=u$ (a single point) and $(x+r)\cdot\overline{Z}_p=0$. The point u separates p and r, and $u\mathcal{E}Q$ from condition (b). Then $G_x\subset G-\overline{Z}_p$ and hence $y\neq p$. Similarly $y\neq r$. Let U_y be a neighborhood of y such that $(x+p+r)\cdot\overline{U}_y=0$. As $y\mathcal{E}G^2$ there is a neighborhood $V_y\subset U_y$ such that $F(V_y)=u_1+u_2$. From the fact that H is an irreducible continuum it follows that both u_1 and u_2 separate p and r in G. Then there exists a neighborhood Y_{pi} $(i=1, 2)$ of p such that $F(Y_{pi})=u_i$ and thus $u_i\mathcal{E}Q$ from (b). The connected set G_x contains a point in V_y and a point x non-$\mathcal{E}V_y$, so either u_1 or u_2 belongs to G_x. But $G_x\cdot Q=0$ and $u_i\mathcal{E}Q$. This is a contradiction and hence $H-Q=0$.

From this it follows that $H=Q=G$. Then $p+r=G^1$ and $G-p-r=G^2$. By a result due to Urysohn and Menger* the set G is an arc with end points p and r.

8. Lemma. *If* $p\mathcal{E}M^2$ *and* $\dim_p M^2>0$, *then* p *is a cut point im kleinen of* M.

There exists a neighborhood U_p such that if $V_p\subset U_p$ then

$$(2) \qquad F(V_p)\cdot M^2 \neq 0,$$

and

$$(3) \qquad F(V_p) \supset \text{at least two points}.$$

There exists a sequence of neighborhoods $V_1, V_2, \cdots$ of p such that $V_i\subset U_p$, $\overline{V}_{i+1}\subset V_i$, $d(V_i)<1/i$, $F(V_i)=u_i+v_i$. Consider the set $\overline{V}_{i-1}-V_i$. Obviously the set $u_{i-1}+v_{i-1}+u_i+v_i=G\subset\overline{V}_{i-1}-V_i$. Let $x\mathcal{E}\overline{V}_{i-1}-V_i$. Then $\mathcal{C}(x, \overline{V}_{i-1}-V_i)\cdot G\neq 0$, for otherwise there would be a separation of $\overline{V}_{i-1}-V_i$ into mutually separated sets containing $\mathcal{C}(x, \overline{V}_{i-1}-V_i)$ and G respectively. But this separation would effect a separation of M, which was connected. Then $\overline{V}_{i-1}-V_i$ contains at most four components. But $\mathcal{C}(u_i, \overline{V}_{i-1}-V_i)$ must contain u_{i-1} or v_{i-1} for otherwise $V_i+\mathcal{C}(u_i, \overline{V}_{i-1}-V_i)$ is a neighborhood $V_p \subset U_p$, but $F(V_p)=v_i$ contrary to (3). Similarly for $\mathcal{C}(v_i, \overline{V}_{i-1}-V_i)$. Also $\mathcal{C}(u_{i-1}, \overline{V}_{i-1}-V_i)\cdot(u_i+v_i)\neq 0$, for otherwise $V_{i-1}-\mathcal{C}(u_{i-1}, \overline{V}_{i-1}-V_i)$ is a neighborhood V_p and (3) is not true. Similarly for $\mathcal{C}(v_{i-1}, \overline{V}_{i-1}-V_i)$. Thus $\overline{V}_{i-1}-V_i$ has at most two components and each contains a point of u_i+v_i and a point of $u_{i-1}+v_{i-1}$. From (2) we have that $u_i \mathcal{E} M^2$ or $v_i \mathcal{E} M^2$. Further at least one of these points is a point of M^2 and is such that dim $M^2\cdot(\overline{V}_{i-1}-V_i)>0$ at the point. For suppose $u_i+v_i \subset M^2$, dim $M^2\cdot(\overline{V}_{i-1}-V_i)=0$ at both points. There exists a neighborhood $U_{u_i} \subset V_{i-1}$ such that $F(U_{u_i})\cdot M^2\cdot(\overline{V}_{i-1}-V_i)=0$, and similarly a neighborhood $U_{v_i} \subset V_{i-1}$. Then $W_p=V_i+U_{u_i}+U_{v_i}$ is a neighborhood of p such that $W_p \subset U_p$ and $F(W_p)\cdot M^2=0$ contrary to (2). In case $u_i \mathcal{E} M^2$, v_i non-$\mathcal{E} M^2$, $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)=0$, we define $W_p=V_i+U_{u_i}$. The only other possibilities are the interchange of u_i and v_i.

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* P. Urysohn, loc. cit., and K. Menger, loc. cit., p. 303.

Case I. Suppose (a) $u_i \mathcal{E} M^2$, (b) $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$, (c) v_i non-$\mathcal{E} M^2$. Since $u_i \mathcal{E} M^2$ there is a neighborhood $U_{u_i} \subset V_{i-1}-\overline{V}_{i+1}-v_i$ such that $F(U_{u_i})=x+y$. Since there is a component of $\overline{V}_i-V_{i+1}$ containing u_i and a point of $u_{i+1}+v_{i+1}$, either x or y, let us suppose y, belongs to V_i-V_{i+1}. Then $x \mathcal{E} M^2$, since $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$. And $\mathcal{C}(x, \overline{U}_{u_i}\cdot(\overline{V}_{i-1}-V_i))=H$ is a continuum such that $u_i+x \subset H^1$. Now if W_{u_i} is any neighborhood of u_i such that x non-$\mathcal{E} W_{u_i}$, then $W_{u_i}\cdot U_{u_i}$ is a neighborhood such that $F(W_{u_i}\cdot U_{u_i})\cdot(\overline{V}_{i-1}-V_i) \subset F(W_{u_i})\cdot H$. Then from (b) we have that $F(W_{u_i})\cdot H\cdot M^2\neq 0$. Thus by the lemma of §7 the continuum H is an arc from x to u_i. Now let N_i denote u_i plus all points of $\overline{V}_{i-1}-V_i$ that can be joined to u_i by an *open* arc-segment of $\overline{V}_{i-1}-V_i$, i.e., a point $z \mathcal{E} \overline{V}_{i-1}-V_i$ is a point of N_i if there is an arc $A \subset \overline{V}_{i-1}-V_i$ with end points z and u_i such that $M-A+u_i+z$ is closed. Evidently $H \subset N_i$. Since u_i is a point of order 1 of $\overline{V}_{i-1}-V_i$, we see that N_i is an arc or homeomorphic with an arc minus one end point. Consider the second possibility. As V_i+N_i is a neighborhood of p contained in U_p, it follows from (2) and (c) that one point of $\overline{N}_i-N_i$ is a point $q \mathcal{E} M^2$. Since M is locally connected at q we have that q is the only point of $\overline{N}_i-N_i$. Then N_i+q is an arc and q may be joined to u_i by the open arc-segment N_i-u_i. Thus $q \mathcal{E} N_i$ which is a contradiction. Hence N_i is an arc and let u_i and q be its end points. As V_i+N_i-q is a neighborhood of p, $q \mathcal{E} M^2$ from (2) and (c). If q non-$\mathcal{E} u_{i-1}+v_{i-1}$, there exists a neighborhood $U_q \subset V_{i-1}-\overline{V}_i$ such that $F(U_q)$ consists of just two points. Then just as was the case with H, we may show that $\overline{U}_q-N_i+q$ is an arc by using the lemma of §7. Then any point of this new arc belongs to N_i by definition, a contradiction.

Hence $q \mathcal{E} u_{i-1}+v_{i-1}$, say $q=u_{i-1}$. As $q \mathcal{E} M^2$ and q is a limit point of $M-\overline{V}_{i-1}$, $q=u_{i-1}$ is a point of order 1 of $\overline{V}_{i-1}-V_i$. Hence N_i is a component of $\overline{V}_{i-1}-V_i$ and $\overline{V}_{i-1}-V_i-N_i$ is closed. Then

$$\overline{V}_{i-1} - V_i = N_i + M_i,$$

where M_i is a continuum containing v_i and v_{i-1}.

Case II. Suppose (a) $u_i+v_i \subset M^2$, (b) $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$, (c) $\dim_{v_i} M^2\cdot(\overline{V}_{i-1}-V_i)=0$. From (c) there exists a neighborhood U_{v_i} such that $\overline{U}_{v_i}\subset V_{i-1}-\overline{V}_{i+1}-u_i$ and $F(U_{v_i})\cdot(\overline{V}_{i-1}-V_i)\cdot M^2=0$. The proof in Case II is exactly the same as Case I except that where a neighborhood of p is formed by taking V_i plus some open set, we take $V_i+U_{v_i}$ plus the open set.

Case III. Suppose (a) $u_i+v_i\subset M^2$, (b) $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$, (c) $\dim_{v_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$. Let N_{u_i} and N_{v_i} denote the sets consisting of u_i and v_i respectively together with all points of $\overline{V}_{i-1}-V_i$ that can be joined to u_i or v_i, as the case may be, by an arc of $\overline{V}_{i-1}-V_i$ such that the arc-segment is an open subset of M. The sets N_{u_i} and N_{v_i} are either arcs or homeomorphic with an arc minus one end point. Either N_{u_i} or N_{v_i} is an arc, for otherwise $V_i+N_{u_i}+N_{v_i}$ is a neighborhood of p and we have a contradiction exactly as in Case I. Also we may prove, similar to the proof of Case I, that one of these must be an arc with u_{i-1} or v_{i-1} as one end point and this arc is a component of $\overline{V}_{i-1}-V_i$ and also an open subset of it. Hence

$$\overline{V}_{i-1} - V_i = N_i + M_i, \quad N_i\cdot M_i = 0,$$

where each is a continuum joining a point of u_i+v_i to a point of $u_{i-1}+v_{i-1}$.

Thus we have seen in any case that each set $\overline{V}_{i-1}-V_i$ consists of two components N_i and M_i, and suppose the components are so lettered that $N_i\cdot N_{i+1}\neq 0\neq M_i\cdot M_{i+1}$. Then

$$\overline{V}_1 = \left(p + \sum_{i=2}^{\infty} N_i\right) + \left(p + \sum_{i=2}^{\infty} M_i\right)$$

is the sum of two continua having only p in common. Hence p is a cut point im kleinen of M.

9. Theorem. *If M is any continuum, then $M^2=H+K$, where* (a) *H is vacuous or* dim $H=0$, (b) *if $p\in H$ then* $\dim_p M^2=0$, (c) *K is vacuous or a countable set of arcs A_i*, (d) *each arc-segment A_i is an open subset of M*, (e) *$A_i\cdot A_j=0$, $A_i\subset A_j$, or $A_j\subset A_i$.*

Let K denote the set of all points p such that $p\in M^2$ and $\dim_p M^2>0$. Then if $q\in M^2-K$, $\dim_q M^2=0$. Then as $H=M^2-K\subset M^2$, $\dim_q H=0$ for each $q\in H$. Now let $p\in K$. By the lemmas of §§6 and 8, there is an arc B_p one of whose end points is p such that $B_p\subset M^2$ and $<B_p>$, i.e. B_p minus its end points, is an open subset of M. And the arcs B_p may be chosen so that if x is any point of an open arc-segment of M, there is some point $p\in K$

such that $x \mathcal{E} <B_p>$. Then the set $[<B_p>]$ for all points $p \mathcal{E} K$ is a set of open subsets of M covering all points of open arc-segments of M. By the Lindelöf property there is a countable subset, $<B_1>$, $<B_2>$, $\cdots$, which covers the same set. Since each $B_i \subset M^2$ it follows that if the sum of any finite number of the B_i's is connected, then it is an arc. From this we find that

$$N = \sum_i B_i$$

consists of a countable number of maximal connected subsets, N_1, N_2, $\cdots$, each of which is homeomorphic with a closed, half-open, or an open interval. Take the interval $(0, 1)$ and let Φ_i be the homeomorphism which carries this interval (closed, half-open, or open) into N_i. There are three cases.

Case I. N_i is homeomorphic with the closed interval. In this case N_i is an arc and let $A_{i1} = N_i$, $A_{ij} = 0$ for $j > 1$.

Case II. N_i is homeomorphic with the half-open interval $(0, 1>$. In case $\lim_{n \to \infty} \Phi_i(n/(n+1))$ exists and is a point $x_i \mathcal{E} M^2$, then $N_i + x_i$ is an arc $\subset M^2$ and we define $A_{i1} = N_i + x_i$, $A_{ij} = 0$ for $j > 1$. In case the limit does not exist or x_i non-$\mathcal{E} M^2$, we define $A_{ij} = \Phi_i(I_j)$, where I_j is the closed interval $(0, j/(j+1))$.

Case III. N_i is homeomorphic with the open interval $<0, 1>$. Suppose (a) $\lim_{n \to \infty} \Phi_i(1/n)$ exists and is a point $x_i \mathcal{E} M^2$, (b) $\lim_{n \to \infty} \Phi_i(n/(n+1))$ exists and is a point $y_i \mathcal{E} M^2$. Then $N_i + x_i + y_i$ is an arc $\subset M^2$ and we define $A_{i1} = N_i + x_i + y_i$, $A_{ij} = 0$ for $j > 1$. If (a) is true and (b) false, we define $A_{ij} = x_i + \Phi_i(I_j)$, where I_j is the half-open interval $<0, j/(j+1))$. If (b) is true and (a) false, we define $A_{ij} = y_i + \Phi_i(I_j)$, where I_j is the half-open interval $(1/(j+1), 1>$. If both (a) and (b) are false, we define $A_{ij} = \Phi_i(I_j)$, where I_j is the closed interval $(1/(j+2), (j+1)/(j+2))$.

We shall show now that $[A_{ij}]$ is the required countable set of arcs, i.e., for every i and j, $A_{ij} \subset K$, and if $p \mathcal{E} K$, then $p \mathcal{E} A_{ij}$ for some i and j. The first part is obvious from the definition of A_{ij}. Now if $p \mathcal{E} K$, there is an arc $B_p \subset M^2$ with p as one end point such that $<B_p>$ is an open subset of M. Now as $[B_i]$ covers all such open subsets, there exists an integer i such that $<B_p> \subset N_i$. If $p \mathcal{E} N_i$, then $p \mathcal{E} A_{ij}$ for some j. If p non-$\mathcal{E} N_i$, since p is an end point of B_p, it follows that either $\lim_{n \to \infty} \Phi_i(1/n)$ or $\lim_{n \to \infty} \Phi_i(n/(n+1))$ exists and is the point p. In this case $p \mathcal{E} A_{i1}$.

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Reprinted from the
TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY
Vol. 40, No. 1, pp. 24-36
July, 1936

PERIODIC ONE-PARAMETER GROUPS IN THREE-SPACE*

BY

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Introduction. We shall consider a one-parameter group of homeomorphisms of euclidean three-space, here designated by E, into itself. We shall postulate for our group a certain weak periodic character with respect to the points of E and shall show that it is closely related to the rotation group of three-space. In particular it will be easy to determine when it is topologically equivalent to that group. In anticipation of the explanatory remarks of the next section, we may formulate our principal theorem as follows:

A continuous one-parameter group $T(x; t)$,

$$x \epsilon E, \qquad -\infty < t < \infty,$$

of pointwise-periodic homeomorphisms of E into itself, whose period function is bounded in every sphere, is a topological quasi-rotation group. In particular, it is the topological rotation group whenever the period function is constant over the moving points.

1. To make clear the meaning of the theorem, let us denote by R a fixed euclidean three-space of reference, given in cylindrical coordinates: (r, θ, z). Let $F(r, z)$ be any positive continuous function defined for all z and all $r>0$. For a fixed real number t, the correspondence

$$T_F(r, \theta, z; t): \qquad (r, \theta, z) \rightarrow (r, \theta + 2\pi[F(r, z)]t, z)$$

defines a homeomorphism of R into itself which we shall call a quasi-rotation. The totality of these mappings, for all real t, is a group and we call it a quasi-rotation group. In particular, when $F(r, z)$ is constant, it is the rotation group and is independent, topologically speaking, of the value of the constant. Now let H denote any homeomorphism of R into E. Corresponding to each t, for a fixed $F(r, z)$, H induces a homeomorphism of E into itself: their totality, for all t, will be called a *topological quasi-rotation group* of E. The proof of our theorem will consist in constructing a suitable function F and homeomorphism H.

2. In accordance with our premises, let $T(x; t)$ be a *one-parameter group* of homeomorphisms of E such that

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* Presented to the Society, September 12, 1935; received by the editors June 12, 1935.

† National Research Fellow.

(i) $T(x; t)$ is continuous simultaneously in x and t;

(ii) to each solid sphere S in E there corresponds an integer N such that if $x \epsilon S$ there exists at least one $\bar{t} = \bar{t}(x)$ for which

$$T(x; \bar{t}) = x, \qquad 0 < \bar{t} < N.$$

For each x the set of points $T(x; t)$ is a continuous image of the set of real numbers. It will be called the *orbit* of x and designated by O_x. The set of real numbers t for which $T(x; t) = x$ is a closed subgroup (not the identity) of the real additive group, depending on x, and O_x is homeomorphic to the corresponding factor group. Therefore O_x is either a simple closed curve, the homeomorph of a circumference, or the single point x. When $O_x = x$, x will be called a *fixed point* and said to have the period zero. Otherwise x will be called a *moving point* and its period $p(x)$ will be the least positive number for which $T[x; p(x)] = x$. By condition (ii) the period function $p(x)$ is bounded in every sphere. By the group property it is constant over each orbit. The following lemma is quite easy.

Lemma 1. *If the sequence of points x_n converges to a moving point x, and the sequence of numbers $p(x_n)$ to a number p, then p is a positive integral multiple of $p(x)$.*

As an immediate corollary $p(x)$ is lower semi-continuous.

3. **The orbit-space E^*.** If the point y is in O_x, there is a t, $0 \leqq t < p(x)$, such that $T(x; t) = y$. It is a trivial consequence of the group property that two orbits have no point in common unless they coincide. Further, if any sequence of points x_n converges to a point x, then

$$\overline{\sum O_{x_n}} \supset O_x$$

by condition (i), and

$$\overline{\sum O_{x_n}} - \sum O_{x_n} \subset O_x$$

in virtue of the boundedness of the period function. That is, $O_x = \lim O_{x_n}$. We may conclude that if the points x_n converge to the point at infinity, that is, properly diverge, then any sequence of points drawn from the corresponding orbits must properly diverge. It is convenient to adjoin to E the point at infinity, P_∞, thus obtaining a closed three-sphere $\overline{E}$. We shall extend $T(x; t)$ by the following definition: $T(P_\infty; t) = P_\infty$ for all t. In this way P_∞ becomes its own orbit and is a fixed point. The extended function remains continuous simultaneously and it is still true that if $x_n \rightarrow x$, then $O_{x_n} \rightarrow O_x$.

We can now construct the important auxiliary space E^* whose points are in 1–1 correspondence with the orbits of $\overline{E}$, and which we may suppose metrized.† The continuous single-valued correspondence of $\overline{E}$ into E^* we shall denote by L. If a is any point of E^*, the inverse $L^{-1}(a)$ is an orbit, and A is a closed set in E^* when $L^{-1}(A)$ is closed. Finally since E is compact, connected, and locally connected, E^* will have all of these properties. It will be proved later that E^* is *homeomorphic to a closed 2-cell*. We may now define for every a in E^* a function $p^*(a)$ which has as its value the constant value assumed by $p(x)$ on $L^{-1}(a)$. The function $p^*(a)$ is lower semi-continuous.

4. Let F denote the set of fixed points of $\overline{E}$, M the set of moving points, D the subset of M at which $p(x)$ is discontinuous, and C the subset of M where $p(x)$ is continuous. Then $\overline{E}=F+C+D$, these sets being mutually exclusive. It is clear that F is closed, and it is not vacuous since it certainly contains P_∞. We shall assume that F does not exhaust E, as otherwise the analysis of $T(x; t)$ is trivial. Then $M=C+D$ is open and not vacuous. It is clear that $E^*=L(F)+L(C)+L(D)$, the three sets being mutually exclusive.

LEMMA 2. *The set C is open and not vacuous.*

This is an easy consequence of Lemma 1. There is no reason, a priori, why D should be vacuous, but we shall see later that it is.

The system of orbits contained in M is readily shown to be a regular family as defined by H. Whitney.‡ It is in fact a very special type of such families and we shall therefore be free to exploit the "cross sections" which he has introduced to obtain what shall here call a *true section*.

Definition. If A is a subset of E^*, any relatively closed subset K of $L^{-1}(A)$ which *cuts each orbit of* $L^{-1}(A)$ *exactly once* will be called a true section of $L^{-1}(A)$. It is clear that K will be homeomorphic to A.

By Lemma 2, $D+F$ is closed; it follows that $L(D+F)$ is closed and that its complement $L(C)$ is open in E^*.

LEMMA 3. *If b is any point in* $L(C)$ *there is a neighborhood U of b such that* $L^{-1}(U)$ *contains a true section.*

Let U_1 be any neighborhood of b contained in $L(C)$. The orbits filling $L^{-1}(U_1)$ constitute a regular family, so that if q is any point of $L^{-1}(b)$ there exists a set S which is a *cross section through* q of the curves filling $L^{-1}(U_1)$ [in the sense of Whitney]. Let R_n be the closed sphere of center q and radius $1/n$. The set $R_n\cdot S$ is a cross section through q.§ It will now be shown that for some n each orbit contains at most one point of $R_n\cdot S$. Suppose, contrariwise, that in each set $R_n\cdot S$ there is a point q_n such that O_{q_n} cuts $R_n\cdot S$ in a second point r_n. There is then for each n a number t_n such that $|t_n| \leqq \frac{1}{2}\, p(q_n)$ and such that $T(q_n; t_n) = r_n$. Since $p(x)$ is bounded in every bounded set, it may be assumed that $\{t_n\}$ converges to a unique limit t_0. The sequences $\{q_n\}$ and $\{r_n\}$ converge to q, so that $q = \lim r_n = \lim T(q_n; t_n) = T(q; t_0)$. On account of the continuity of $p(x)$ and the fact that $|t_n| \leqq \frac{1}{2} p(q_n)$ it follows that $t_0 = 0$. Consider now the arcs $c_n d_n$ made up of points $T(q_n; t)\,[-|t_n| < t < |t_n|]$. These arcs contain q_n and r_n. But if $q'q''$ is any arc of O_q containing q, and λ is any number, it is easy to see that for some n, there is an arc $c_n d_n$ contained in an arc of the λ-neighborhood (in Whitney's sense, loc. cit.) of $q'q''$. Since this is in contradiction with his cross section, we may conclude that there must exist an n_1 such that each orbit which cuts $R_{n_1} \cdot S$ cuts it exactly once.

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† See Hausdorff, *Mengenlehre*, 1927, p. 145, for a definition of distance between closed sets.

‡ Annals of Mathematics, vol. 34 (1933), pp. 244–270.

§ We shall not repeat the definition and properties of the Whitney cross sections, except by implication.

The set $R_{n_1} \cdot S$ is mapped in a 1–1 manner on the set $L(R_{n_1} \cdot S)$. Because $R_{n_1} \cdot S$ is compact this mapping is a homeomorphism. From the properties of the Whitney section the set $L(R_{n_1} \cdot S)$ must contain all points in some neighborhood U of b. If K is the part of $R_{n_1} \cdot S$ which is mapped into U, K is a true section of $L^{-1}(U)$.

5. In this section we shall investigate the nature of the inverse images of certain simple sets in E^* preparatory to showing that E^* is a closed 2-cell.

LEMMA 4. *If $\alpha\beta$ is an arc in $L(C)$, then $L^{-1}(\alpha\beta)$ contains a true section and is homeomorphic to the product of a circle and a segment.*

About each point b of $\alpha\beta$ choose a neighborhood U such that $L^{-1}(U)$ contains a true section K. Any subarc $\alpha_1\beta_1$ of $\alpha\beta$ which lies in a single U is clearly such that $L^{-1}(\alpha_1\beta_1)$ contains a true section $\alpha_1'\beta_1'$. Since $\alpha\beta$ may be covered by a finite set of the U's mentioned above, there is on $\alpha\beta$ a set of non-overlapping arcs $(\alpha_0\alpha_1)$, $(\alpha_1\alpha_2)$, $\cdots$, $(\alpha_{n-1}\alpha_n)$ whose sum is $\alpha\beta$ and each of which is in a single U. Let $\alpha_{i-1}''\alpha_i'$ be a true section of $L^{-1}(\alpha_{i-1}\alpha_i)$. There is a real number t_1 such that $T(\alpha_1''; t_1) = \alpha_1'$. Consider the arc $\alpha_0''\alpha_1' + T(\alpha_1''\alpha_2'; t_1)$.† This arc is a true section of $L^{-1}(\alpha_0\alpha_1 + \alpha_1\alpha_2)$ obtained by turning the true section of $L^{-1}(\alpha_1\alpha_2)$ so that its first end point matches with the last end point of the true section of $L^{-1}(\alpha_0\alpha_1)$. By making successive turnings of this type, we obtain an arc $\alpha'\beta'$ which is a true section of $L^{-1}(\alpha\beta)$.

If I is the unit interval and C_1 is a circle of circumference 2π we can show that $L^{-1}(\alpha\beta)$ and the product-space $I x C_1$ are homeomorphic as follows. Let $h(x)$ be any homeomorphic mapping between $\alpha'\beta'$ and I. Suppose now that y is a point of $L^{-1}(\alpha\beta)$ which is on the orbit cutting $\alpha'\beta'$ in x and suppose that $T(x; t) = y$ where $0 \leqq t < p(x)$. Then because of the continuity of $p(x)$ in C, a homeomorphism between $L^{-1}(\alpha\beta)$ and $I x C_1$ is defined by the correspondence: $y \longleftrightarrow h(x) x 2\pi t/p(x)$.

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† This shall denote the set of all points x such that $x = T(x'; t_1)$ for some $x' \subset \alpha_1''\alpha_2'$.

Lemma 5. *If the arc $\alpha\beta$ is in $L(C)$ except for the point α which is in $L(F)$, then $L^{-1}(\alpha\beta)$ has a true section and is a closed 2-cell.*

Express $\alpha\beta-\alpha$ as $\sum_{n=1}^{\infty}\sigma_n$ where the σ_n's are consecutive closed arcs, σ_n and σ_{n+1} abutting at a common end point β_n. Let S_n be a true section of $L^{-1}(\sigma_n)$. The end points b_1 and b_1' of S_1 and S_2 respectively, corresponding to β_1, lie on the same orbit. Choose a t^* so that $T(b_1'; t^*)=b_1$. The arc $S_2'=T(x; t^*)$, x ranging over S_2, is also a true section of $L^{-1}(\sigma_2)$, and S_1+S_2' is a true section of $L^{-1}(\sigma_1+\sigma_2)$. Continuing this indefinitely we obtain a true section K of $L^{-1}(\alpha\beta-\alpha)$. It is clear (compare §3) that if f denotes the fixed point $L^{-1}(\alpha)$, $K+f$ is closed and is the desired section. That $L^{-1}(\alpha\beta)$ is a 2-cell is now trivial.

Lemma 6. *If J is a simple closed curve in $L(C)$, then $L^{-1}(J)$ contains a true section and is a torus.*

We express J as the sum of two arcs $\alpha\gamma\beta$ and $\alpha\gamma'\beta$. Let acb and $a'c'b'$ be true sections of $L^{-1}(\alpha\gamma\beta)$ and $L^{-1}(\alpha\gamma'\beta)$ respectively. Choose t and t' so that

$$T(a; t) = a', \qquad T(b; t') = b'.$$

Let h be a continuous function defined for all x of acb which has the value t at a and t' at b. The set of points

$$T[x; h(x)], \qquad x \text{ in } acb,$$

is a true section of $L^{-1}(\alpha\gamma\beta)$, and is an arc whose end points coincide with a' and b' respectively. Together with the arc $a'c'b'$ it gives the desired true section of $L^{-1}(J)$. That $L^{-1}(J)$ is a torus can now be seen by defining a homeomorphism similar to the one used in Lemma 4.

6. We shall devote this section to proving that D is vacuous. A certain type of set used below will first be described. Let B be the product of a circle and an interval, in other words a bounded cylinder. Let points of the base be specified by an angular coordinate θ $(0\leqq\theta\leqq 2\pi)$ and on one base of the cylinder identify sets of k points of the form $\theta+2n\pi/k$ $(n=0, 1, \cdots, k-1)$. Any homeomorph of this figure is called a twisted strip of index k.

Lemma 7. *Let $\alpha\beta$ be an arc which, with the exception of α, lies in $L(C)$; and which is such that as x approaches α along $\alpha\beta$, $p^*(x)$ approaches $kp^*(\alpha)$. Then $L^{-1}(\alpha\beta)$ contains a true section and is a twisted strip of index k.*

We observe first that $L^{-1}(\alpha\beta)$ is locally connected. For let g be any point of this set, U_ϵ the open sphere about g of radius ϵ. There is an open arc of the orbit O_g which contains g and lies entirely in U_ϵ. Let $\delta<\epsilon$ be such that the sphere U_δ about g does not intersect O_g except in points of the open arc. Let $\gamma=L(g)$. It is clear, by §3, that there is an arc σ, open in $\alpha\beta$ and containing γ, such that every orbit in $L^{-1}(\sigma)$ intersects U_δ. Let Q be the subset of $L^{-1}(\sigma)$ consisting of all points x such that $x=T(\bar{x};\bar{t})$, where $\bar{x}$ is some point of U_δ and $T(\bar{x};t)$ is in U_ϵ for every t, $0\leqq t\leqq\bar{t}$ if $t>0$, and for every t, $0\geqq t\geqq\bar{t}$ if $t<0$. It is now easy to see that Q is connected and open in $L^{-1}(\alpha\beta)$. Hence $L^{-1}(\alpha\beta)$ is locally connected.

Therefore if q is any point of $L^{-1}(\alpha)$ and S is a cross section through q of the regular family filling $L^{-1}(\alpha\beta)$, S is locally connected.† Then there must be an arc qr which is contained in S and which has q and no other point on $L^{-1}(\alpha)$.

It will now be shown that qr cuts no orbit more than once. For, if it did, it would contain a proper subarc $r'r$ with the same property. Now $r'r\subset L^{-1}(\alpha'\beta)$, where $\alpha'\beta$ is a proper subarc of $\alpha\beta$. In virtue of Lemma 4, a quite elementary argument will show that there exists a point v of $r'r$ and there exists an arbitrarily small arc σ on the orbit of v, containing v as inner point, such that on arbitrarily near orbits (possibly on O_v itself) there exist arcs arbitrarily near to σ (in the sense of Whitney) which meet $r'r$ (therefore S) in at least two distinct points. Hence the first condition that S be a cross section (Whitney, loc. cit.) fails at the point v. Therefore qr cuts all orbits at most once.

It follows that qr is mapped homeomorphically on a subset $\alpha\beta_1$ of $\alpha\beta$ and $\alpha\beta_1$ is an arc. If β_1 is different from β, there is an arc $\beta_1'\beta'$ which is a true section of $L^{-1}(\beta_1\beta)$. By turning $\beta_1'\beta'$ so that β_1' coincides with r we obtain an arc which we denote by qr' and which is a true section of $L^{-1}(\alpha\beta)$.

That the surface swept out by qr' under the action of the group is a twisted strip of index k may be seen as follows. Let Q be a cylinder which is the product of K and I, K being a circle of circumference 2π and I being a unit interval. Let y be any point of $L^{-1}(\alpha\beta)$ and denote by x' the point in which the orbit through y cuts qr'. Let t be such that $T(x';t)=y[0\leqq t<p(y)]$. Let $h(x')$ be a homeomorphism carrying qr' into I. Now consider the following correspondence. If y is in $L^{-1}(\alpha\beta-\alpha)$, y corresponds to $h(x')x2\pi t/p(y)$. If y is in $L^{-1}(\alpha)$, y corresponds to $h(x')x2\pi t/(kp(y))$. This determines a correspondence between $L^{-1}(\alpha\beta)$ and certain points of Q. However, when the proper identifications are made on Q to make it a twisted strip of index k, the above correspondence actually is a homeomorphism of $L^{-1}(\alpha\beta)$ and this twisted strip.

PRINCIPAL LEMMA 1. *The set D is vacuous; that is, $p(x)$ is continuous on the moving points.*

We shall show that $L(D)\subset E^*$ is vacuous. We know that $L(D)$ is closed in $L(M)$, hence $L(D)=P+G$ where P is perfect in $L(M)$ and G is countable. There are two cases:

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† Whitney, loc. cit.

(i) P is vacuous. Here $L(D)=G$ and contains an isolated point b. Since $L(F)$ is closed, there exists in E^* an arc $\beta'b$ which, except for b, belongs to $L(C)$ where $p^*(x)$ is continuous. Since D is, in this case, a one-dimensional F_σ,† it is well known that its complement in E is arcwise connected. Therefore if f^* denotes any point of $L(F)$, there is in E^* an arc $\beta'f^*$ which contains no point of $L(D)$. Let f be the first point of this arc which is in $L(F)$. In the sum of the arcs $b\beta'$ and $\beta'f$ there is an arc $b\beta f$ such that the open arc belongs to $L(C)$.

Now it is quite easy to see that, at an isolated point b in $L(D)$, $p^*(x)$ approaches a unique limit, say $kp^*(b)$, where by Lemma 1, k must be an integer. Then from previous lemmas, $L^{-1}(b\beta)$ is a twisted strip of index k, and $L^{-1}(\beta f)$ is a 2-cell. It is clear that $L^{-1}(b\beta f)$ is homeomorphic to a complex which contains no absolute cycle, but which is a cycle mod k. Such a configuration cannot exist in E, as can be seen from the Alexander Duality Theorem.‡ In this first case we have now arrived at a contradiction.

(ii) P is not vacuous. The function $p^*(a)$ considered relatively to P is lower semi-continuous and there must be a point b in P at which $p^*(a)$ is continuous§ considered relatively to P. Let O be an open set in $L(M)$ which contains b and which is such that if $a \epsilon O$, $p^*(a) < Np^*(b)$ where N is some fixed integer. Assume further that O is so chosen that if $a \epsilon O \cdot P$,

$$(1) \qquad | p^*(a) - p^*(b) | < \frac{p^*(b)}{6N}.$$

Let H_i^* be the set of all points a in $L(C)$ such that

$$(2) \qquad | ip^*(b) - p^*(a) | < \frac{p^*(b)}{6}.$$

Let $H_i = H_i^* \cdot O$. Assume now that O is chosen so small that it may be written as a sum of (mutually exclusive) sets as follows: $O=P\cdot O+G\cdot O+\sum_1^N H_i$. Because b is in $L(D)$ there must be an integer $k \geqq 2$ such that H_k is not vacuous. Let B denote all those points of the boundary of O_k which lie in O, where O_k is used to denote a component of H_k. Let x_0 be a point in $L^{-1}(O_k)$ and let x_1 and x_2 be two points on distinct orbits in $L^{-1}(O\cdot P)$. There are at least two such points because P is dense in itself. Join x_0 and x_1 by an arc x_0x_1 which is in $L^{-1}(O)$ and which intersects neither $L^{-1}(G)$ nor x_2 (the set to be avoided is a 1-dimensional F_σ). Join x_0 and x_2 by an arc x_0x_2 which is in $L^{-1}(O)$ and which intersects neither $L^{-1}(G)$ nor the arc previously constructed† (except of course at the point x_0). Let x_3 be the first point of $L^{-1}(P\cdot O)$ on x_0x_1 and let x_4 be the first point of $L^{-1}(P\cdot O)$ on x_0x_2. These points are in $L^{-1}(B)$. The set $L(x_0x_3)+L(x_0x_4)$ is connected and locally connected; let $L(x_3)=a_3$ and $L(x_4)=a_4$. There must be an arc a_3a_4 which is a subset of $L(x_0x_3)+L(x_0x_4)$. Except for a_3 and a_4 this arc lies entirely in O_k. The function $p^*(a)$ considered along the arc a_3a_4 must approach an integral multiple of $p^*(a_3)$, say $k_3p^*(a_3)$, as a approaches a_3 along the arc, and as a approaches a_4 along the arc, $p^*(a)$ must approach an integral multiple of $p^*(a_4)$, say $k_4p^*(a_4)$.

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† By the "Summensatz": Menger, *Dimensionstheorie*, p. 92.

‡ Alexander, these Transactions, vol. 23, p. 333; and Pontrjagin, Annals of Mathematics, vol. 35, p. 904.

§ Hausdorff, *Mengenlehre*, 1927, p. 255.

For a in the interior of the arc we must have, since $a\epsilon O_k\subset H_k\subset H_k^*$,

$$|\,kp^*(b)-p^*(a)\,|<\frac{p^*(b)}{6}\cdot \tag{3}$$

Since a_i $(i=3,4)$ is in $O\cdot P$ and since $k\leqq N$, (1) implies

$$|\,p^*(b)-p^*(a_i)\,|<\frac{p^*(b)}{6k}\cdot$$

This inequality implies

$$|\,kp^*(b)-kp^*(a_i)\,|<\frac{p^*(b)}{6}\cdot \tag{4}$$

From (3) and (4) we have the following:

$$|\,p^*(a)-kp^*(a_i)\,|<\frac{p^*(b)}{3}\cdot \tag{5}$$

For a on the arc, $p^*(a)>p^*(b)$ and therefore (5) implies that k_3 and k_4 both equal k.

Let a_5 be some point on the interior of a_3a_4. The sets $L^{-1}(a_5a_3)$ and $L^{-1}(a_5a_4)$ are both twisted strips of index k. These strips combine to form a set which is a cycle modulo k but which contains no cycle in the absolute sense. Again by the Alexander Duality Theorem such a set cannot be imbedded in $\bar{E}$ and a contradiction has been reached. Therefore both P and G must be vacuous and the set $L(D)$ is vacuous.

7. It will assist us in analyzing F to construct a certain homeomorphism, defined as follows: For any point x, $H(x)$ is the point $T[x;\,p(x)/2]$. That $H(x)$ is a homeomorphism follows from the properties of $T(x;\,t)$ and the fact

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† Again the set to be avoided is a 1-dimensional F_σ.

that $p(x)$ is continuous at moving points. If x_0 is a fixed point the continuity of $H(x)$ at x_0 follows even though $p(x)$ is not continuous at x_0 on account of the boundedness required of $p(x)$. The fixed points of $H(x)$ are identical with the fixed points of $T(x; t)$. The transformation $H(x)$ clearly has period 2.

LEMMA 8. *The set F is nowhere dense.*

The set F is closed and would contain inner points if the lemma were false. But if the fixed points of H contain inner points, H is the identity† and this case has been excluded from consideration.

LEMMA 9. *If O is an open connected set which is invariant under $T(x; t)$, then $O - O\cdot F$ is arcwise connected.*

Suppose that $O - O\cdot F = A + B$ where A and B are disjoined non-vacuous sets such that $A\overline{B} + \overline{A}B = 0$. Since $O - O\cdot F$ is open, A and B must be open. The set O is transformed homeomorphically into itself by H, and furthermore $H(A) = A$, $H(B) = B$, and $H(O\cdot F) = O\cdot F$. A new transformation K of O into itself will now be defined. If $x \epsilon B + O\cdot F$, $K(x) = H(x)$; if $x \epsilon A$, $K(x) = x$. The transformation K is a homeomorphism of A into itself. But the set of fixed points of K contains inner points. Since K has period 2 and since O is locally euclidean, it follows from Newman's theorem that every point of O is fixed under K. This contradicts the definition of K and hence $O - O\cdot F$ must be connected. That it is arcwise connected is now immediate, since $O\cdot F$ is closed in O.

It follows at once from this lemma that M is connected.

LEMMA 10. *If q is any fixed point there is a connected open set O including q which has an arbitrarily small diameter and is invariant.*

Take a sphere containing q and let O be the set swept out by this sphere under the action of $T(x; t)$. The conditions on $p(x)$ imply that the diameter of O may be made small by properly choosing the sphere about q.

LEMMA 11. *Every fixed point is arcwise accessible from M.*

This is a consequence of the two preceding lemmas.

PRINCIPAL LEMMA 2. *The set $L(M)$ in E^* is homeomorphic to the euclidean plane.*

Let J denote any simple closed curve of $L(M)$. The set $L^{-1}(J)$ is a torus by Lemma 6, and $\overline{E} - L^{-1}(J)$ is the sum of two connected domains O_1 and O_2, say, such that $\overline{O}_i - O_i = L^{-1}(J)$, $i = 1, 2$, by the Alexander Duality Theorem. Since $L^{-1}(J)$ is invariant, it is clear that O_1 and O_2 are invariant. By Lemma 9,

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† See Newman, Quarterly Journal of Mathematics, vol. 2 (1931), p. 1.

$O_i - O_i \cdot F$ is connected. It is clear that $L(O_1 - O_1 \cdot F)$ has no point in common with $L(O_2 - O_2 \cdot F)$ and that $L(M) - J$ is equal to the sum of these two sets; that is, J separates $L(M)$. We have, in fact, shown the stronger result that J separates E^*. Now if G is any arc of J, $L^{-1}(G)$ is a closed proper subset of $L^{-1}(J)$. Therefore $\bar{E} - L^{-1}(G)$ is open and connected, and $\bar{E} - L^{-1}(G) - F$ is connected. But $L[\bar{E} - L^{-1}(G) - F] = L(M) - G$. Therefore G does not separate $L(M)$.

To complete the proof it is necessary to show† that one of the domains of $L(M) - J$ is compact in $L(M)$ and that the other is not. One of the domains, we may suppose it is O_2, contains P_∞. It is clear that $L(O_2 - O_2 \cdot F)$ is not compact in $L(M)$. Then we have left to show that O_1 does not contain points of F: in this event $L(O_1)$ *is* compact in $L(M)$. Since J is accessible‡ from each of its domains, if $L(O_1)$ contains a point of $L(F)$, E^* must contain an arc $a_1 a_0 a_2$ such that $a_1 + a_2$ is in $L(F)$, a_0 is in J, the open arc $a_1 a_0$ is in $L(O_1 - O_1 \cdot F)$, and the open arc $a_0 a_2$ is in $L(O_2 - O_2 \cdot F)$. The sets $L^{-1}(a_1 a_0)$ and $L^{-1}(a_2 a_0)$ are 2-cells by Lemma 5 and it is clear that their sum is a 2-sphere S_2 which has in common with the torus $L^{-1}(J)$ the simple closed curve $L^{-1}(a_0)$. By the Duality Theorem, S_2 separates E into exactly two open connected domains G_1 and G_2. Now $L^{-1}(J - a_0)$ is connected and must lie entirely in one or the other of these sets, say in G_1. This implies that there cannot be points of G_2 in both O_1 and O_2. Since the points of $L^{-1}(a_1)$ and of $L^{-1}(a_2)$ (points of S_2) must be accessible from G_2, this is a contradiction.

8. We shall now prove the following lemma:

LEMMA 12. *If $\alpha\beta$ is an arc in $L(M)$ and if qr and $q'r'$ are two true sections of $L^{-1}(\alpha\beta)$, there exists a real continuous function $h(x)$ defined everywhere on qr such that $T[x; h(x)]$ is a homeomorphic mapping of qr on $q'r'$.*

Let a denote any point of $\alpha\beta$, x and x' the corresponding points of qr and $q'r'$. It is clear that the correspondence $x' \longleftrightarrow x$ engenders a homeomorphism.

For a definite point x let t be a real number such that $T(x; t) = x'$. Let x_n denote any sequence of points of qr converging to x. Assign to each x_n a number t_n, $|t_n| \leqq \frac{1}{2}p(x_n)$, such that $T(x_n; t + t_n) = x_n'$. At least one such number exists and at most two. We may suppose our sequence of points such that t_n converges to a limit t'. Then $T(x_n; t + t_n) \rightarrow T(x; t + t')$ which is equal to $T(x'; t')$. On the other hand $x_n' \rightarrow x'$ so that $T(x; t') = x'$, from which we may conclude that $t' = mp(x')$ where m is some integer or zero. Since $p(x)$ is continuous at x', t' cannot exceed $\frac{1}{2}p(x')$ in absolute value, and m must be zero. It follows that for almost all x_n only one choice of t_n is possible and we have seen that this choice must be continuous at x. It can now be seen that there must exist a sufficiently small subarc $q''r''$ of qr containing x as an inner point such that a real continuous function $h''(x)$ may be defined on it for which $T[x; h''(x)]$ maps $q''r''$ homeomorphically on the corresponding subarc of $q'r'$. We need only choose an arc so small that the numbers t_n assigned above are actually *less* in absolute value than one-half the corresponding period.

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† L. Zippin, *On continuous curves*, American Journal of Mathematics, vol. 52 (1930), pp. 331–350.

‡ L. Zippin, loc. cit.

The arc qr may be expressed as a sum of non-overlapping subarcs $q_0q_1, q_1q_2, \cdots, q_{k-1}q_k$, such that there is defined on $q_{i-1}q_i$ a function $h_i(x)$ of the required sort. Now the values of h_1 and h_2 at q_1 can differ only by an integral multiple m of the period at that point. Then $h_2' = h_2 + mp(x)$ is a new function defined on q_1q_2 which is of the required type, and this new function agrees with h_1 at q_1. The function which is equal to h_1 on q_0q_1 and equal to h_2' on q_1q_2 is of the required type over $q_0q_1+q_1q_2$. We can continue in this manner, and after k steps we arrive at a function defined over qr which satisfies the lemma.

LEMMA 13. *If S is a closed 2-cell in $L(M)$ there exists a true section of $L^{-1}(S)$.*

We can express S as the sum of a finite number of closed 2-cells $A_1, \cdots, A_n$, of small diameter such that $A^i = \sum_1^i A_j$ is itself a closed 2-cell and the intersection of A^k and A_{k+1} is an arc J_{k+1}. Since $L^{-1}(S)$ admits true sections locally by Lemma 3, it may be assumed that to each A_i there corresponds a true section B_i of $L^{-1}(A_i)$.

Let $B^1 = B_1$, and suppose we have constructed a true section B^k of $L^{-1}(A^k)$. Let J be the image of J_{k+1} in B^k and J' its image in B_{k+1}. Let $h(x)$ be the function defined on J' in the preceding lemma, and let $H(x)$ be any continuous extension of this function over all of A_k. Then $B^{k+1} = B^k + T[x; H(x)]$ where x is allowed to vary over B_{k+1}. This set is seen to be a true section of $L^{-1}(A^{k+1})$. The induction is complete and Lemma 13 is proved.

It follows from this lemma that $L^{-1}(S)$ can be mapped homeomorphically into the product of a circular disk D and a circle C in such a way that the path curves of $L^{-1}(S)$ are carried into the sets dxC where $d \epsilon D$. Hence if S is a closed 2-cell in M and a is a point of S, $L^{-1}(a)$ cannot bound in $L^{-1}(S)$.

9. The fact just proved will be useful in the following lemma:

LEMMA 14. *If a is a point of $L(M)$, $L^{-1}(a)$ cannot bound in M.*†

If $L^{-1}(a)$ bounds in M, it must bound a singular 2-chain K which is contained in M. The transformation L carries K into a closed compact subset of $L(M)$ which we may suppose to be contained in a closed 2-cell S where

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† This is very similar to a theorem due to Seifert, Acta Mathematica, vol. 60, p. 224.

$S \subset L(M)$. But now $L^{-1}(a)$ bounds in $L^{-1}(S)$, which from the remark above cannot happen.

Now since any simple closed curve bounds in $\overline{E}$ even after the removal of a finite number of points, it is a corollary to the lemma that F is an *infinite* set.

LEMMA 15. *For every point z in F, $F-z$ is connected.*

If z is a point for which the lemma is false, F may be expressed as the sum of two closed (in F) sets X and Y whose intersection is a subset of z (possibly vacuous). If x and y denote points of X and Y respectively and if $L(x)=a$ and $L(y)=b$, there is an arc ab in E^* which contains no points of $L(F)$ other than a and b. This follows from the accessibility of points of F. Now $L^{-1}(ab)$ is a 2-sphere and bounds in $\overline{E}-z$ by the Alexander Duality Theorem. On the other hand if a_1 is an inner point of ab, $L^{-1}(a_1)$ bounds in the complement of X and in the complement of Y. This situation, however, contradicts corollary W^i of Alexander (loc. cit.), and this contradiction establishes the lemma.

Since F contains infinitely many points and is connected on the removal of any one of them, it must be connected.

PRINCIPAL LEMMA 3. *$L(\overline{E})$ is homeomorphic to a closed half-plane.*

If we let x and y denote any two distinct points of F, and if a, b, and ab are used as in the preceding lemma, the 2-sphere $L^{-1}(ab)$ separates E. Since it is an invariant surface we can conclude that the arc ab separates E^*. A proper subset of this arc gives rise to a proper subset of $L^{-1}(ab)$ and this does not separate E. We conclude that the arc ab separates E^* *irreducibly*.† But it is further clear that each domain of $E-L^{-1}(ab)$ contains at least one orbit, and since this bounds in its domain, we see that each domain must contain fixed points. Therefore the points x and y must separate the set F, and the points a and b must separate $L(F)$. Therefore $L(F)$ is a continuum without cut points which is separated by every pair of its points. It follows that $L(F)$ must be a simple closed curve.‡ Finally we have the fact that E^* is a closed 2-cell with $L(F)$ as its boundary.

It is clear that $E^*-L(P_\infty)$ which is identically $L(E)$ is a closed half-plane, the points $L(F)$ corresponding to the boundary line.

10. We are in a position to bring our entire argument to a close. We shall choose a Cartesian coordinate system (z, r) in $L(E)$, $-\infty<z<\infty$, $0\leq r<\infty$, so that $L(F)$ is the z-axis, $r=0$. This is possible by the preceding

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† L. Zippin, *Characterization of the closed 2-cell*, American Journal of Mathematics, vol. 55 (1933), pp. 207–217.

‡ R. L. Wilder, *Concerning simple continuous curves*, American Journal of Mathematics, vol. 53 (1931), pp. 39–55; Corollary, p. 48.

lemma. Now $L(M)$ is the point set $r>0$. We can obviously express this set as the sum of a countable number of "rectangles" such that the sum of any finite number of consecutive rectangles is a closed 2-cell. Corresponding to each of these there is a true section in M, by Lemma 13, and these can be "fitted" to each other by the construction in that lemma, where the induction must be carried indefinitely. Therefore M possesses a true section M', and M' is homeomorphic to $L(M)$. The homeomorphism is given by the function L. The function L is certainly a homeomorphism between F and $L(F)$, since each point of F is its own orbit. Then it is clear that L maps $M'+F$ homeomorphically into $L(E)$. For if a sequence of points of $L(M)$ converges to a point of $L(F)$, the corresponding orbits must converge to the fixed point in E, and therefore the corresponding points of M', which lie on these orbits, must converge to it.

We may now transfer to $E''=M'+F$ the coordinate system (z, r). We shall extend this to a cylindrical coordinate system in all of E. Consider now an arbitrary point x of E, our original three-space. It belongs to an orbit O_x which intersects E'' in a single point, say x'', with coordinates (z'', r''), say. If $r''=0$, x'' is a fixed point and $x=x''$. Then the "θ-coordinate" of x shall be indeterminate. If $r''>0$, there is a unique t'', $0\leqq t''<p(x'')=p(x)$, such that $T(x'', t'')=x$. The θ coordinate of x shall be $2\pi t''/p(x'')$. Conversely, it is clear that any three numbers z, θ, r,

$$-\infty<z<\infty, \qquad 0\leqq\theta\leqq 2\pi,$$

determine a unique point of E. With this coordinate system established in E we may take E to be the "reference-space" R of §1, and the homeomorphism of that section to be identity. The function $F(z, r)$ is now $1/p(z, r)$, $p(z, r)$ being the value of the period function at the point (z, r) of E''. We have seen that this function is independent of θ and is continuous at the moving points, i.e., points for which $r>0$.

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TOPOLOGICAL GROUP FOUNDATIONS OF RIGID SPACE GEOMETRY

BY

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Dedicated to the memory of Bella Zippin, mother of one and friend of both of us.

1. Hilbert, after building up geometry from a point of view which relegated continuity considerations to the background [4], built up plane geometry afresh [5] on the foundation of groups of *homeomorphisms* of the *number plane*, both "continuity" concepts. It is this latter point of view with which we are concerned in this paper. Hilbert carried out this program only for the plane but he hinted that it might be possible to carry it out in a somewhat similar way for three-space. Kerékjártó took up the problem [6] for three-space and made a great deal of progress with it, but, as he wrote before the recent developments in topological groups, he found it necessary to employ a stronger set of axioms than is necessary now.

Relying on the theory of topological groups we recently characterized the rotation group of three-space [9], and in commenting on that work P. A. Smith [12] suggested that it might provide the basis for an extension of Hilbert's program to three-space.

The purpose of this paper is twofold. In the first place we shall characterize the classical space geometries on the basis of a fairly weak set of axioms, and in the second place we shall show that Hilbert's axioms for the case of the plane can be weakened by replacing what might be called his "three-point condition" by a two-point condition. We achieve this latter purpose more or less incidentally to the first.

In comparing our set of three-space axioms with Hilbert's axioms for the plane we find that the first axiom is the same in both cases. The third axiom of this paper is weaker than Hilbert's, and our second purpose above is to show that this weaker axiom also suffices in Hilbert's case. Our second axiom is weaker than Hilbert's second axiom in that it relates to the subgroup leaving a single point fixed, but it is incomparably stronger in what it asks of that one subgroup.

We do not, in this paper, settle the question of whether or not Hilbert's second axiom is adequate for three-space geometries. This question is bound up with an unsolved problem concerning transformation groups.

Finally we wish to remark that instead of assuming that the space we are dealing with is ordinary three-space, it is only necessary to make certain topological assumptions on the space from which it follows by virtue of the

Presented to the Society, October 28, 1939; received by the editors December 20, 1939.

same group axioms that the space is actually a number-space. But we reserve discussion of this matter for another occasion. In this connection see the abstract by the authors in the Bulletin of the American Mathematical Society, vol. 45 (1939) (no. 349).

2. **The axioms.** We formulate two sets of axioms, the first set for the plane, and the second set for three-space. The first set, to which we proceed immediately, is the set used by Hilbert except that it has been materially weakened in the manner described.

We assume then that there is given a system (E_2, G) where E_2 is the number-plane and G is a set of sense preserving homeomorphisms of this plane, and that this set satisfies the following conditions:

2.1. *The system G is a group.*

The assumption tacitly implicit in the above is that G is effective, that is, that no element except the identity leaves all of E_2 fixed.

With each point x in space there exists a subgroup G_x consisting of elements of G which leave x fixed.

2.2. *If x is any point of E_2 and y is distinct from x, then $G_x(y)$ is infinite.*

This axiom could be reformulated so that it would be entirely analogous to our axiom 2.2′ for three-space but we do not carry this out. It would involve almost no change in the work.

2.3. *Let (x, y) and (x', y') be two pairs of points of E_2, where the points of a pair are not necessarily distinct. If there exist pairs (x_n, y_n) and (x_n', y_n'), the first arbitrarily near (x, y), the second arbitrarily near to (x', y'), and if there exists an element of G taking (x_n, y_n) to (x_n', y_n'), then there exists an element of G taking (x, y) to (x', y').*

As we have said, this set of axioms is exactly Hilbert's except that the third axiom has been weakened to a condition on pairs instead of triples of points.

We now formulate our axioms for three-space. We assume that there is given a system (E_3, G) where E_3 is ordinary three-space and G is a set of sense preserving homeomorphisms of E_3 satisfying the following conditions:

2.1′. *The same as 2.1.*

2.2′. *There exists a point p of E such that the group G_p is a proper subgroup of G and for a sequence of points p_n approaching p the sets $G_p(p_n)$ are at least two dimensional.*

2.3′. *The same as 2.3.*

Occasionally we shall refer to the situation described by the first set of axioms as the *plane case* and to the situation described by the second set as the *space case*. In both cases we shall prove that ordinary geometric concepts such as "line" and "distance" (and in the space case "plane") may be defined in terms of G in such a way that we obtain either euclidean or hyperbolic geometry and that G is the group of rigid motions of the corresponding geometry we obtain. We do this in the plane case by proving Hilbert's axioms [5], that is, by proving that the two-point condition 2.2 implies the three-point condition. The space case we treat in detail and show in detail that there are only the two systems.

Our approach to this problem differs from Hilbert's in one important respect. Hilbert analyzes more or less directly the topological nature of the orbit $G_p(x)$. In three-space this course seems to us not feasible until much more is known about strongly homogeneous subsets of space. But even granting such knowledge our procedure has the advantage of making available the results of topological group theory. Thus, we proceed at once to a study of the group G_p as a topological transformation group. In brief summary, we first confine our attention to a suitable invariant neighborhood of the point p where orbits under G_p can be proved compact. We form the effective group in this neighborhood, and show that any sequence of elements of this group has a subsequence which converges to a homeomorphism of the neighborhood into itself. We then augment our group by the addition of such homeomorphisms. It transpires, only considerably later, that this enlargement is an illusory one. The enlarged group is then shown to be a compact topological transformation group on a "locally euclidean" space. From our previous work, we then know that our orbits are necessarily manifolds, and it is not difficult to show that they are indeed spheres. From an earlier paper of ours we learn also the complete structure of the group and its behavior in the neighborhood. We are now in a position to show, by an argument patterned on one of Hilbert's, that the neighborhood above coincides with space.

The use of a "two-point" rather than "three-point" axiom shows itself in one or two interesting ways in the study of G_p but becomes a matter of considerable moment in the study of the group G as a topological transformation group of the space. This argument is given in §12. To this point the case of the plane or of space may be treated more or less simultaneously, and it seems to us, in fact, that much of this generalizes with no great difficulty to four-space and perhaps farther.

The remainder of the paper is devoted to a study of the geometry induced in space by the group G. Here, after a few paragraphs, we are on ground already explored by Kerékjártó. His paper was not known to us until our own had been completed, and we carry out the program essentially as we had it. We do this in part for completeness sake, in part because the form in which our solution is set differs sufficiently from Kerékjártó's. At one point in proving the linearity of our planes we borrow a very ingenious idea from Kerékjártó's paper which shortens considerably an argument of our own.

3. In a great part of the paper we treat the two cases simultaneously, calling the space (which of course is either E_2 or E_3) simply E. When we speak of a sphere or a rotation group we of course mean the one appropriate to the dimension of the space.

Let p be a point of E, which for E_2 may be any point, but which for E_3 is to be the point specified by 2.2′.

Hilbert points out that for any x the set $G_p(x)$ is closed. This is a consequence of 2.3. Thus: let x_n be a sequence of points in $G_p(x)$ converging to a limit point x_0. There are elements g_n in G_p such that $x_n = g_n(x)$. The element g_n takes the pair (p, x) to the pair (p, x_n). By 2.3 there is a g in G which takes (p, x) to (p, x_0) and this element certainly is in G_p.

By a similar method it is seen that $G(x)$, which ultimately will be shown to coincide with E, is closed.

4. *Let R be the set of points x such that $G_p(x)$ is compact.* This set is not vacuous for it contains p.

LEMMA 1. *The set R is open.*

Let x be any point in R, and let B be a conditionally compact open set containing $G_p(x)$ in its interior. Let S be the boundary of B. We assert that *there is an open set V containing $G_p(x)$ such that G_p carries no point of S inside V.* Otherwise there must exist a set of elements g_n in G_p and a set of points b_n in S such that at least one point of the set $g_n(b_n)$ lies in every neighborhood of the compact set $G_p(x)$. There is no loss in assuming that b_n approaches some point b in S and $g_n(b_n)$ approaches a point a in $G_p(x)$. But then there must be an element of G_p taking b to a which means that a point of S is in $G_p(x)$ contrary to the choice of B.

Let W denote those components of the above determined V which include points of $G_p(x)$. The set W is open and no element of G_p carries a point of W outside of B. For such an element would also leave an element of W inside B (any point namely in which W meets $G_p(x)$), and it would therefore carry some point of W into S which is impossible.

It has now been shown that all points of W have orbits inside B. Therefore every point of W has a compact orbit and x is in an open set W all of whose points have compact orbits as the lemma demands.

4.1. The proof shows even more than is required in Lemma 1. It shows, for any point x in R, that x is an interior point of a set W, such that $G_p(W)$ is a conditionally compact set. If $\overline{W}$ is the closure of W, then $G_p(\overline{W})$ is compact and x is seen to be an interior point of a set $\overline{W}$ such that $G_p(\overline{W})$ is compact. These facts together with the Heine-Borel theorem enable us to state the following lemma.

LEMMA 2. *If M is any compact subset of R, then $G_p(M)$ is compact.*

5. We consider now the action of G_p on R. Conceivably G_p has a non-

trivial subgroup leaving all of R fixed. Later this possibility will be ruled out, but meanwhile we must take account of it. Let G_p^1 be the subgroup leaving all of $\overline{R}$ fixed. This subgroup includes the identity at any rate, and G_p/G_p^1, which will be denoted for brevity by H, is an effective transformation group of $\overline{R}$.

Our next task is to show that H may be extended to become a compact transformation group of $\overline{R}$. It will be assumed that E is assigned a bounded metric, say the metric of a three-sphere or a two-sphere according to the case, which is obtained by adding a point to E. This means that we can define a distance between any two transformations of E into itself or between any two transformations of a subset of E into itself. For example if f and g are two transformations of $\overline{R}$ into itself

$$d(f, g) = \text{l.u.b. } d[f(x), g(x)]$$

where x ranges over $\overline{R}$. Under this distance H becomes a metric space.

5.1. LEMMA 3. *If a sequence of elements of G converges everywhere on a set M to a limit h, then h is continuous on M. If M is compact the convergence is uniform.*

Let us prove first that h is continuous. Let m be any point of M and let S be any sphere with $f(m)$ as a center. Let m_i be a sequence of points of M approaching m. We shall show that almost all the points $h(m_i)$ are inside or on S. Assume that this is not true for an infinite subsequence, say m_{k_i}, and let $m_{k_i}m$ be a short arc joining m_{k_i} to m. Since g_n approaches h and since $h(m_{k_i})$ is outside S by assumption, there will certainly be an integer, say n_i, such that $g_{n_i}(m_{k_i})$ is outside S. We may assume without loss of generality that every $g_{n_i}(m)$ is inside S. Hence there is a point, say y_{k_i}, on the arc $m_{k_i}m$ such that $g_{n_i}(y_{k_i})$ is on S. Assume that $g_{n_i}(y_{k_i})$ approaches a point b on S. Then g_{n_i} takes the pair (y_{k_i}, m) which is near (m, m) to the pair $[g_{n_i}(y_{k_i}), g_{n_i}(m)]$ which is near $[b, h(m)]$. This is a contradiction from which the continuity of h follows.

We shall next show that the convergence is uniform in case M is compact. If the convergence is not uniform, there is for some positive number e an infinite set of indices $k_1, k_2, \cdots$ and a set of points $m_1, m_2, \cdots$ in the set M such that

$$d[g_{k_i}(m_i), h(m_i)] \geqq e.$$

There is no loss in assuming that the sequence m_i converges to m and that $g_{k_i}(m_i)$ converges to a point b. From the above inequality b and $h(m)$, which is the limit of $h(m_i)$ by the continuity, are distinct. The transformations g_{k_i} therefore take the pair (m_i, m) which is near (m, m) to the pair $[g_{k_i}(m_i), g_{k_i}(m)]$ which is near $[b, h(m)]$. By 2.3 there must be an element of G taking m to each of the distinct points b and $h(m)$. This contradiction shows that the convergence is uniform.

LEMMA 3.1. *Let g_n be a sequence of elements of G converging to h everywhere on a set M. Then if m_i approaches m it follows that $g_i(m_i)$ approaches $h(m)$.*

The set B containing the points m and all m_i's is compact. Hence on this set g_n converges uniformly to h. Let e be any positive number. For all n greater than an integer N_1,

$$d[g_n(x),\ h(x)] < e/2$$

for all x in B. Since h is continuous, there will be an integer N_2 such that if n is greater than N_2

$$d[h(m_n),\ h(m)] < e/2.$$

For all n greater than N_1 and N_2 we have not only this latter inequality but we have as a consequence of the first inequality

$$d[g_n(m_n),\ h(m_n)] < e/2.$$

The last two inequalities yield the desired conclusion.

LEMMA 3.2. *If a sequence f_n of elements of G_p converges everywhere on a compact set M, invariant under G_p, to a transformation f, then f is a homeomorphism of M into itself.*

In view of Lemma 3 it is only necessary to show that f has a single valued inverse and that $f(M) = M$.

If f does not have a single valued inverse, there must be two distinct points b and c such that $f(b) = f(c)$. Then the elements f_n take the pair (b, c) to the pair $[f_n(b), f_n(c)]$ and by 2.3 there is an element in G which takes both b and c to $f(b)$. This contradiction shows that f is one-one, and since M is compact f must take M homeomorphically to $f(M)$. We know that $f(M)$ is a subset of M, and to complete the proof of the lemma it remains only to show that $f(M)$ coincides with M.

Let b be any point in M. Then there is an element m_n in M such that $f_n(m_n) = b$. Assume that the sequence m_n approaches a point m. Now f_n takes the pair (m_n, m) which is near (m, m) to the pair $[b, f_n(m)]$ which is near $[b, f(m)]$. Hence some element of G takes m to both points b and $f(m)$ which is possible only if $f(m) = b$.

5.2. LEMMA 4. *The group-space H defined in §5 is conditionally compact.*

Yet Y be a countable dense subset of R. Let g_n be an infinite sequence of elements of H. Strictly speaking the g_n's are not elements of G_p but there are elements of G_p coinciding with these elements on $\overline{R}$, and properties of the group G_p may be used in examining the g_n's.

For any point y in Y the sequence $g_n(y)$ is conditionally compact and has a convergent subsequence. The limit of this sequence belongs to $G_p(y)$ and is a point of R. By the diagonal process there exists a subsequence f_n of the elements g_n such that $f_n(y)$ converges to a unique point of R for *every* element y of Y. Then, on Y, the sequence f_n converges to a pointwise limit function f.

It will now be shown that f is uniformly continuous in every conditionally *compact* open subset R_1 of R. In order to do this it must be shown that for every positive e there exists a positive d such that whenever y and y' of $Y \cdot R_1$ are nearer to each other than d, the corresponding $f(y)$ and $f(y')$ are nearer than e. If this is not the case, there must exist in Y a sequence of points y_n and y_n' which may be supposed to converge to the same point z of $\overline{R}_1$ such that $f(y_n)$ and $f(y_n')$ also converge and converge to two distinct points y and y' in $\overline{R}_1$. This means that some of the elements g_n take a pair of points near z to a pair near (y, y'). By 2.3 there is an element of G carrying the point z to the two points y and y'. This is manifestly impossible and the contradiction establishes the uniform continuity of f on the set $R_1 \cdot Y$.

This uniform continuity of f permits us to extend it, and we assume it is so extended, to a single valued continuous transformation of R (which of course is locally compact) into itself. The sequence f_n, which originally was known to converge only on Y, is now seen to converge everywhere on R to the transformation f. By Lemma 3 the convergence is uniform on compact sets which implies that the sequence f_n of elements of H must be a Cauchy sequence because our metric brings two functions close which agree closely outside of a neighborhood of "infinity."

5.3. When we speak of a topological transformation group we use the term with the definition as given in our papers referred to in the bibliography.

LEMMA 5. *The group H may be extended to a compact group H which is an effective topological transformation group of R.*

The space H is conditionally compact so that if the space is made complete the resulting space $\overline{H}$ will be compact. The preceding lemmas and their proofs show that Cauchy sequences of H will actually converge to homeomorphisms of R into itself. The space $\overline{H}$ is therefore a transformation group of R. To be sure that it is a topological transformation group we must prove that if g_n approaches g, these being elements in $\overline{H}$, and x_n approaches x in R then $g_n(x_n)$ approaches $g(x)$. This follows as in the proof of Lemma 3.1.

That $\overline{H}$ is an effective group follows from the fact that distinct elements of the space $\overline{H}$ arise from nonequivalent Cauchy sequences and these give rise to distinct limiting transformations.

6. The present section contains some simple, purely group theoretical, considerations which will be of use to us later.

THEOREM 1. *The only two dimensional manifolds which are coset spaces (orbits) of a compact connected group H are the two-sphere, torus, and projective plane.*

There is no loss in assuming that H is effective in its action on the manifold so that H is a Lie group [10]. The theorem then follows from the corresponding theorem on Lie groups due to Cartan [2].

We note without giving the proof, which is not difficult, the following: If a circle group operates on a torus and has a fixed point, then it must leave every point of the torus fixed.

THEOREM 2. *The only compact group of sense preserving transformations which can act effectively and transitively on a two-sphere M is the group of rigid rotations of the two-sphere.*

For connected groups the theorem is true [9]. If H is the group, let H^* be the identity component of H, and let x be a point of the sphere. The dimension of $H^*(x)$ is the same as the dimension of $H(x)$, namely two, and therefore $H^*(x)$ coincides with M. Since H^* is connected it must be the two-sphere rotation group. Assuming H^* is not all of H means that, for some y, $(H^*)_y$ is a proper subgroup of H_y because every element of H, being sense preserving, has a fixed point; the connectedness of the group $(H^*)_y$ (it is of course circular) shows that it is the component of the identity of H_y. Then $(H^*)_y$ is invariant in H_y. Let M^* be the decomposition space of M under $(H^*)_y$. The group $H_y/(H^*)_y$ acts on this space which is an interval. Hence $H_y/(H^*)_y$ contains only the identity, or it is a group of two elements which merely interchanges the ends of M^* while leaving a "middle" point fixed. The latter possibility cannot occur, for if it did H_y would contain an element moving y. Hence $H_y/(H^*)_y$ contains only the identity element and $(H^*)_y$ is not a proper subgroup of H_y as we assumed. The contradiction shows that H^* coincides with H and that H is the group of rigid rotations of the two-sphere.

COROLLARY. *The only compact group of sense preserving transformations of three-space into itself with at least one two dimensional orbit is the two-sphere rotation group.*

7. In the present section we confine our attention to the space case. The groups H and G_p have the same orbits in R, and since these orbits are closed $\overline{H}$ has the same orbits as do H and G_p. Let H^* be the component of the identity of $\overline{H}$. The orbit of a point under H^* will have the same dimension as the orbit of the point under $\overline{H}$. Hence H^* has a sequence of two dimensional orbits approaching the fixed point p. We will now consider the action of the effective compact transformation group H^* on the connected locally "euclidean" space R^0, where R^0 denotes that component of R which contains p. (It is conceivable that some subgroup of H should leave all of R^0 fixed. We assume without changing our notation that this is not the case. There is no loss in this process as we might as well have assumed we were working with R^0 before.)

It follows from theorems on the structure of coset spaces [10] that any two dimensional orbit of H^* in R^0 must be a two dimensional manifold. Any two dimensional orbit of H^* in R^0 must be, therefore, one of three types of manifold, the two-sphere, the projective plane, or the torus. The projective plane cannot be imbedded in E, so that the number of possibilities is reduced immediately to two, the sphere and torus.

In the orbit space associated with R^0, call it R^*, every two dimensional orbit is a cut point of order two precisely. The set of such orbits is open [10]. The space R^* has one non-cut point, at least, namely the orbit consisting of the point p only. By the cyclic element theory, R^* must be either a line, a ray, or an interval. It cannot be a line because it contains a non-cut point. It cannot be an interval for this would mean that R^0, an open subset of E, would be compact. Hence R^* is a ray, and this shows that *all orbits* in R^0, with the exception of p, are two dimensional orbits which *are either spheres or tori.* The group $\overline{H}$ can be seen to be a Lie group because it operates on a locally euclidean connected space with locally connected orbits [10]. It will now be of dimension three at most [10], and it will be effective on each one of its two dimensional orbits. For, if a subgroup left all of a manifold orbit fixed this same subgroup would leave the whole space fixed by a simple application, as in an earlier paper of ours [8], of a theorem of Newman. If all two dimensional orbits are spheres, then H^* is the rotation group of a sphere, for this is the only connected compact effective transformation group of a sphere. If all two dimensional orbits are tori, then H^* is a two dimensional toral group, for this is the only Lie group which can be effective on a torus.

It is intuitively clear that all orbits must be spheres and we now give the proof. We will show that if one orbit is toral then all orbits are. Assume that one orbit $H^*(x)$ is toral. If H^* is two dimensional, then H^* is a toral group and all orbits are tori. If H^* is three dimensional there must be a circular subgroup K leaving x fixed. But if a circular subgroup leaves one point of a torus fixed it must leave every point fixed. Hence K leaves all of $H^*(x)$ fixed and, since this separates space, we see by a familiar device that K leaves all of space fixed. In this case H^* is not effective. We are therefore led to conclude that all two dimensional orbits are tori.

This last situation is impossible. For if H^* is a toral group we can form a true section B of the space, that is, we can find a closed set B which has one and only one point on each orbit in R. The set B will have to be homeomorphic to R^* and will be a ray. Now let $H^*(x)$ be a toral orbit inside a neighborhood U of p which is homeomorphic to three-space. There will be in $U-H^*(x)$ a one-cycle Z which does not bound in $U-H^*(x)$ and which is outside $H^*(x)$; that is, it is contained in the component of $U-H^*(x)$ which does not contain p. Then using the true section B we may deform $H^*(x)$ to the point p. The cycle Z certainly bounds in $U-p$, contradicting its choice.

Therefore not every orbit is a toral orbit and H^* must be the rotation group of three-space, and every two dimensional orbit must be a two-sphere. Let K be a circular subgroup of H^*. There will have to be precisely two points on each two-sphere orbit left fixed by K. These two points will define for us a double valued function everywhere on R^*. The end point of the ray R^* is an exception when the function is single valued. But at any rate it is possible to pick out of these functional values a true section B of the entire space. The existence of the ray B proves R^0 homeomorphic to euclidean three space.

From Theorem 2 of §6 we see that $\overline{H}$ and H^* must coincide, but we can conclude even more.

THEOREM 3. *The group H coincides with $\overline{H}$ and is therefore the rotation group of three-space.*

Let x be any point of R distinct from p. The set $\overline{H}(x)$ is a two-sphere, and $H(x)$ coincides with $\overline{H}(x)$; or in other words H, a subgroup of $\overline{H}$, is transitive on the two-sphere. This is possible only if H is all of $\overline{H}$ [11].

8. In this section we turn to the plane case, falling back on Lemma 5 where we left it.

THEOREM 3.1. *The group H coincides with $\overline{H}$ and is a circle group, the rotation group of the plane. The set R^0 is homeomorphic to a plane.*

The group $\overline{H}$ cannot be totally disconnected, for such a group cannot operate effectively on a locally planar space (as we have shown [8]). Let H^* be the identity-component of $\overline{H}$. As in the space case, the orbits under H^* must be manifolds and therefore simple closed curves: they are obviously one dimensional. It follows that H^* is a Lie group and in particular the circle group [10]. As in the space case the group $\overline{H}$ must operate upon the decomposition space of R^0 by orbits under H^*: this space is a ray, with p as end point, and H^* must be the identity upon it.

Now let g be some element of $\overline{H}$, not the identity. There must be a point x of R^0 such that gx is not x. On the other hand, gx is a point of $H^*(x)$ so that for some g' of H^* we have

$$g'g^{-1}x = x.$$

Since this element is sense preserving and leaves one point of the circle $H^*(x)$ fixed, it leaves all $H^*(x)$ fixed and then all of R^0. Therefore it must be the identity and we conclude that g belongs to H^*. This shows that $\overline{H}$ coincides with H^* and is a circle group. Since H is transitive on $\overline{H}(x)$, it is obvious that H must coincide with $\overline{H}$ which is effective on $\overline{H}(x)$.

Now R^0 is a connected open subset of the plane filled out by a continuous family of simple closed curves and it is intuitively obvious and sufficiently well known that R^0 must be homeomorphic to the plane.

We turn now to a simultaneous consideration of the two cases. What has been done above is summed up in the following theorem.

THEOREM 4. *The group H may be so topologized that it becomes the ordinary* *rotation group of space, and it acts on* R^0, *which is homeomorphic to space, as the ordinary rotation group does—in a properly chosen coordinate system.*

9. THEOREM 5. *The set* R^0 *is closed and so coincides with* E.

The assumption that R^0 is not closed implies that there is an arc px which is contained in R^0 except for its end point which is not in R^0. Since x is not in R^0 (and not in R) there is a sequence of elements g_n of G_p such that $g_n(x)$ tends toward infinity. Let $G_p^{R^0}$ denote all elements of G_p leaving all of R^0 fixed. Under the homeomorphism taking G_p to $G_p/G_p^{R^0}=H$ suppose that g_n goes to $\bar{g}_n$; assume that the sequence $\bar{g}_n$ converges to $\bar{g}$ and that $\bar{g}$ is the image of an element g under this homeomorphism.

Let S_1 and S_2 be two spheres each containing $g(px)$ and such that S_2 lies inside S_1. We may assume that all points $g_n(x)$ are outside S_1. On the arc $g_n(px)$ there are two points $g_n(x_n)$and $g_n(y_n)$, where x_n and y_n are points of px, such that $g_n(x_n)$ is the first point of $g_n(px)$ on S_2 and $g_n(y_n)$ is the first point of $g_n(px)$ on S_1. The points x_n and y_n lie on px in the order px_ny_nx.

There is no loss in assuming that the sequences x_n, y_n, $g_n(x_n)$, and $g_n(y_n)$ converge respectively to x', y', x^* and y^*.

We wish to prove that x' is identical with x. If it is not, x' must be a point of R^0 and the points x_n may also be taken to be in R^0. Hence $g_n(x_n)=\bar{g}_n(x_n)$ converges to $\bar{g}(x')=g(x')$ which is impossible because $g(x')$ is inside S_2. Hence x' and x are identical.

This shows that the points x_n converge to x and of course the points y_n must also converge to x. There are, consequently, elements of G which take the pair (x_n, y_n) which is near (x, x) to the pair $[g_n(x_n), g_n(y_n)]$ near to the district pair (x^*, y^*). There is then an element of G taking x to the two distinct points x^* and y^*. From this contradiction the theorem follows.

COROLLARY. *The group* $G_p^{R^0}=H$, *idle on all of* R^0, *contains only the identity and the group* G_p *is itself the rotation group of space.*

10. THEOREM 6. *The group* G *is transitive on the space* E.

It has already been remarked that $G(x)$ is closed for any x in E. In particular $G(p)$ is closed, and in order to prove our theorem it suffices to prove that $G(p)$ is open.

We know in the plane case as well as in the space case that G_p is a proper subgroup of G and there must be an element g in G and a point q distinct from p such that $g(p)=q$. The set $G_p(q)$ is a sphere. Let q' denote another point of $G_p(q)$ and let t be a varying element of G_p which takes q continuously to q'. There is a neighborhood U of q so small that its translation U' to a neighborhood of q' has no point in common with its original position. The set U' is the image of U under the terminal element of the parameter t.

There is a neighborhood V of p so small that $g(V)$ is inside U. As q is swept continuously to q' it must come in contact with the g-image of every sphere about p and within V.

Now let s be any point of V and let $S=G_p(s)$ be its sphere orbit under G_p. There is a t such that $t(q)$ is in $g(S)$, that is,

$$t\{g(p)\} = t(q) = gg'(s)$$

for some g' in G_p. Then

$$g'^{-1}g^{-1}tg(p) = s.$$

In other words p may be carried to any element of V by some element in G. Therefore p is an interior point of $G(p)$ and every point of $G(p)$ must be an interior point.

11. The fact that G is transitive on E tells us the nature of G_x when x is distinct from p. Let $x=g(p)$. Then $G_x=gG_pg^{-1}$ and G_x is also the rotation group and in a proper system of coordinates "centered" on the point x it acts as the rotation group ordinarily does.

12. Before such geometric concepts as lines and planes can be studied, we need to analyze the nature of G as a topological transformation group. It is in this that we encounter the principal difficulties implicit in our use of the "two-point" form rather than Hilbert's "three-point" axiom. In much of this work we shall continue to treat the plane and space cases together.

LEMMA 6. *Let x_n be a sequence of points converging to x and let g_n in G be such that $g_n(x_n)$ approaches x. Then for any z in E, the set $g_n(z)$ is bounded.*

Let O be the interior of an orbit S of G_x which is so large that it surrounds z, all of the x_n's, and all of the points $g_n(x_n)$. Let O^* be the interior of a larger orbit S^* so that S is interior to O^*. Let zx_n be arcs of O, and suppose now that for infinitely many of the elements g_n it is true that $g_n(z)$ is outside or on S^*. For these n's, and we take it now that all n's are such, the arcs $g_n(x_nz)$ have one point in O^* and one point not in O^*. They therefore have a first point z_n' on S^*. There is a point x_n' on x_nz such that $z_n' = g_n(x_n')$. We may suppose that x_n' converges to a point x' which is in O or S, while z_n' converges to a point z' on S^*. There must be an element g in G which leaves x fixed and carries x' to z'. This is by 2.3 because g_n takes the pair (x_n, x_n') to the pair $[g_n(x_n), z_n']$. However such an element g is in G_x and hence leaves S and its interior invariant so that a contradiction has been reached. Hence $g_n(z)$ is compact because almost all its elements are inside S^*.

The orbit S^* was subject only to the requirement that it surround S so that the following corollary is true.

COROLLARY. *Let x_n be a sequence of points converging to x and let g_n be such that $g_n(x_n)=x_n$. Let z be any point of E. Then any limit point of the set $g_n(z)$ is inside or on any orbit of G_x which includes z and every x_n.*

Our task now is the proof of the following theorem.

THEOREM 6.1. *Let* x_n *approach* x *and let* g_n' *be elements of* G *such that* $g_n'(x_n)$ *approaches* y. *Then there is a subsequence* g_n'' *of the* g_n' *and an element* g^* *of* G *such that* g_n'' *approaches* g^* (*in the sense of pointwise convergence*).

The proof of this theorem is rather long and is based on a number of preliminary lemmas to which we now turn. Lemma 3.1 will also be useful here. As usual we use the letter g for an element of G and we use the letter h for a homeomorphism of E which is not known to be an element of G. Convergence of homeomorphisms, as the statement of the theorem implies, is always taken in this section in the sense of pointwise convergence, that is, h_n converges to h provided that, for each x, $h_n(x)$ converges to $h(x)$.

LEMMA 6.12. *Let* g_n *be a sequence of elements of* G *converging to a homeomorphism* h *of* E. *Let* F *be an arbitrary compact subset of* E *and let a positive number* e *be given. Then there exists an integer* N *such that if* y *and* z *are any two points of* F *for which* $d(y, z) < 1/N$, *and* $n > N$ *then*

$$d[g_n(y), h(z)] < e.$$

The proof of this lemma which is quite similar to various preceding proofs will be omitted.

In the hypotheses of the following lemmas it will frequently occur as it did in the preceding lemma that there is a sequence g_n of elements of G converging to a homeomorphism h of E. From now on we shall express this fact in abbreviated form by writing $g_n \to h$.

LEMMA 6.13. *Let* $g_n \to h$ *and let* i *be a positive integer. Then it is true that* $g_n^i \to h^i$.

The proof will be made by induction. It is true when $i = 1$ and we now assume that it is true for $i - 1$.

Let x be any point of E and let F be the set made up of the points x, $h^{i-1}(x)$, and $g_n^{i-1}(x)$, $(n = 1, 2, \cdots)$. By the hypothesis of the induction F is compact. Let e be any positive number. By the preceding lemma there is an integer N such that if y and z are in F and $d(y, z) < 1/N$ then for $n > N$

$$d[g_n(y), h(z)] < e.$$

On the other hand for sufficiently large n, say n greater than N',

$$d[g_n^{i-1}(x), h^{i-1}(x)] < 1/N.$$

Hence if n is larger than (N, N') we obtain (letting y be $g_n^{i-1}(x)$ and z be $h^{i-1}(x)$)

$$d[g_n^i(x), h^i(x)] < e.$$

Lemma 6.14. *Let* $g_n \to h$ *and let* g *be an arbitrary element of* G. *Then* $gg_n \to gh$.

For an arbitrary x we know that $g_n(x) \to h(x)$. It is then an immediate consequence of the definition of homeomorphism that $gg_n(x) \to gh(x)$.

Lemma 6.141. *Let* $g_n \to h$ *and assume that there are elements* g *and* g' *of* G *such that* $gg_n \to g'$. *Then* $g' = gh$ *and* h *is in* G.

By the preceding lemma $gg_n \to gh$ and hence $g' = gh$.

Lemma 6.15. *Let* K *be a simple closed curve and let* T *be a nonidentical sense preserving homeomorphism with a fixed point. Then there exists a pair of points* x *and* y *of* K *such that* $T^i(x) = x$ *and* $T^i(y) \to x$ *as* $i \to \infty$.

Choose any moving point y. Then $T^i(y)$ must approach monotonically a point x which is fixed.

Lemma 6.151. *Let* $g_n \to h$ *and suppose that, for a definite point* q *of* E, $h(q) = q$. *Then* h *must leave invariant every orbit of* G_q, *that is, for all* x,

$$hG_q(x) = G_q(x).$$

Let x be any point. Since $g_n \to h$ we know that $g_n(x) \to h(x)$ and $g_n(q) \to h(q) = q$. Hence there is an element of G taking q to q and x to $h(x)$. This element is in G_q and we have thus shown that $hG_q(x)$ is in $G_q(x)$. The equality must hold because of the nature of $G_q(x)$ in the two cases.

Lemma 6.152. *Let* x_n *be a sequence of points converging to a point* x *of* E. *Let* g_n *of* G *be such that* $g_n(x_n) \to x$. *Then there exists a homeomorphism* h *of* E *such that* $g_n' \to h$ *for a subsequence* g_n' *of* g_n.

The proof here is similar to that of Lemma 4, but it depends also on Lemma 6.

Lemma 6.16. *Let* $x_n \to x$ *and let* g_n' *be such that* $g_n'(x_n) = y_n \to y$. *Let* g *(in* G*) be such that* $g(y) = x$. *Then* $g_n = gg_n'$ *has a subsequence* g_n'' *which converges to a homeomorphism* h *and* $h(x) = x$.

Since $y_n \to y$, $g(y_n) \to g(y) = x$. Then $g_n(x_n) = gg_n'(x_n) = g(y_n) \to x$. By the preceding lemma there is a subsequence g_n'' and a homeomorphism h such that $g_n'' \to h$. By Lemma 3.1, $g_n''(x_n) \to h(x)$ and hence $h(x) = x$.

We are now ready to prove Theorem 6.1 for the plane and we restate it here for this case.

Theorem 6.1′. *Let* $x_n \to x$ *and let* g_n' *be elements of* G *such that* $g_n'(x_n) \to y$. *Then there is a subsequence* g_n''' *of the* g_n' *and an element* g^* *of* g *such that* $g_n''' \to g^*$. *(For* E_2.*)*

Let g_n'' and h be as in Lemma 6.16. The elements g_n'' are sense preserving and consequently h is sense preserving. For all z distinct from x, $G_x(z)$ is a simple closed curve. Since $hG_x(z)$ is in $G_x(z)$ and since h is a homeomorphism, $hG_x(z) = G_x(z)$ and h is sense preserving on $G_x(z)$ (see Lemma 6.151).

For convenience let $K = G_x(z)$ for an arbitrary point z distinct from x, and let $z' = h(z)$. Since G_x is transitive on K, there is a g' in G_x such that $g'(z') = z$, that is, $g'h(z) = z$. By a previous lemma $g'g_n'' \to g'h$. Since g' is in G_x,

$$g'h(x) = g'(x) = x.$$

It follows that $h' = g'h$ preserves the orbits of G_x and in particular it follows that h' is a sense preserving homeomorphism of K. Moreover $h'(z) = z$.

We wish to prove that h' is the identity on E and we begin by proving that h' is the identity on K.

If h' is not the identity, then there is a pair of distinct points x^* and y^* such that $h'^i(x^*) = x^*$ and $h'^i(y^*)$ approaches x^* as i approaches infinity. For a fixed i we know that $g_n^{*i} \to h'^i$ where $g_n^* = g'g_n'' \to g'h = h'$. In view of these two facts we see that g_n^{*i} for some n and i takes x^* and y^* into a specified neighborhood of x^* which violates the two-point condition. Hence h' is the identity on K.

We shall show next that the set of points of $E - x$ on which h' is the identity contains only inner points. Let z' be any point in $E - x$ which is fixed under h'. Since $h'(z') = z'$, h' must leave invariant the orbits $G_{z'}(z^*)$ for any z^* in E. Let $K' = G_x(z')$. By the argument above we see that h' leaves every point of K' fixed. Let U be a neighborhood of z' so small that points of K' lie outside of U and let V be a neighborhood of z' which is invariant under $G_{z'}$ and is contained in U. For any point z^* of V the simple closed curve $K^* = G_{z'}(z^*)$ must contain a point of K'. This is a fixed point of h' and the argument of the preceding paragraph shows that every point of K^* is fixed under h'. Then it follows that all of V is fixed under h'.

Hence the set of points fixed under h' is both open and closed in $E - x$. This set of fixed points therefore includes $E - x$ and since it is closed it must include E. Hence h' is the identity.

Now $h' = g'h$ so that $h = g'^{-1}$ and h is an element of G_x. This completes the proof of the theorem for the case of the plane. To see this assume for convenience that g_n'' and gg_n' coincide. Then $gg_n' \to h = g'^{-1}$ and $g_n' \to g^{-1}g'^{-1}$.

We turn next to the proof of the theorem for E_3.

THEOREM 6.1''. *Let $x_n \to x$ and let g_n' be elements of G such that $g_n'(x_n) \to y$. Then there is a subsequence g_n'' of g_n' and an element g^* of G such that $g_n''' \to g^*$. (For E_3.)*

Let g_n'' and h be as in the preceding proof, that is, as in Lemma 6.16. The transformations g_n'', and therefore h also, are sense preserving in E. The point x is fixed under h and h preserves orbits $G_x(z)$, these orbits being two-spheres. Let z be a definite point of E distinct from x and let $S = G_x(z)$. Since

h, being sense preserving, has at least one fixed point on S we may assume that z is such a fixed point. Then h will preserve orbits $G_z(y)$ for all y of E.

We are going to look for some simple closed curve K, on S, invariant under G_x and G_z *and* also invariant under h. Let S' denote a sphere orbit under G_z such that some points of S are outside S'. Let F denote the closed intersection of S and S' and let D denote the component of $S-F$ which contains z.

Now S and S' are invariant under h and their intersection F must also be invariant under h. Hence $S-F$ is invariant and D must be invariant under h since it is a component with a fixed point. Let K be the boundary of D on S. It is this set which will be shown to be a simple closed curve. Observe at the moment that K is invariant under h and is a subset of F.

The group G_{xz} leaving x and z fixed is a circle group and F is invariant under G_{xz}. Let y denote any point of K above. Then $K=G_{xz}(y)$. Hence K is a simple closed curve invariant under h. Furthermore h in its action on K must be sense preserving for otherwise it would have to interchange the two components of $S-K$ which it cannot do since we know that D contains a fixed point.

Since G_{xz} is transitive on K there is an element $g'h=h'$ which has all the properties of h which we need to use and in addition has a fixed point on K.

We wish to show that h' is the identity on S ultimately enlarging the set of fixed points to include all of E. By a familiar argument h' must leave all of K fixed. The following lemma will be useful to us as we proceed.

LEMMA 6.17. *If h' leaves fixed a point p and a continuum on an orbit $S^*=G_p(q)$, then h' leaves all of S^* fixed.*

Let F be the set of fixed points of h' on S^*, and let $M=S^*-F$. Let O be a component of M, and let B be the boundary of O. Now from the hypothesis B cannot be zero dimensional, for if it were B would be all of F and so F would not contain a continuum. Hence B is one dimensional, and it must contain a continuum C which contains a point b which is accessible from O. Let U be a neighborhood of b (in S^*) so small that any simple closed curve in U surrounding b must meet C. Let $G_b(y)$ be an orbit so small that its intersection with S^* is in U. Then by an argument given above there is a simple closed curve in the intersection of $G_b(y)$ and S^* which is invariant under h'. Furthermore h preserves sense on this curve. Since this curve surrounds b it must meet C and consequently h' must leave all the curve fixed. As this curve was arbitrarily small we see that b cannot be accessible from O. This is a contradiction which proves the lemma.

We can of course conclude now that h' leaves all of S fixed. Furthermore any small "sphere" with center on S will also be left fixed by h' since it will intersect S in a one dimensional set. The set of "spheres" about x forms a ray (the end point of the ray being of course a point orbit). The above considerations show that the "spheres" of this ray left entirely fixed by h' form a set which is both open and closed. Therefore h' leaves every point of E fixed and is the identity.

Now as in the plane case $h=g'h$ and $h=g'^{-1}$ so that h is in G and indeed in G_{xz}. This completes the proof as in the former case.

12.1. LEMMA 6.2. *If* $g_n \to g$, *then* $g_n^{-1} \to g^{-1}$.

Let x be any point of E. By hypothesis $g_n(x) \to g(x)$. Also $g_n g^{-1}(x) \to gg^{-1}(x) = x$. If we knew that $gg_n^{-1}(x) \to x$ we could conclude that $g_n^{-1}(x) \to g^{-1}(x)$. The proof of the lemma is therefore reduced to the proof of the following special case. *If* g_n *is in* G *and* $g_n(x) \to x$, *then* $g_n^{-1}(x) \to x$.

In order to prove this proposition let S be any sphere about x. We shall show that almost all the points $g_n^{-1}(x)$ are inside S. If this is not true we arrive at a contradiction as follows. For each $g_n^{-1}(x)$ not inside S choose a short arc joining x to $g_n(x)$, call it $xg_n(x)$. Then $g_n^{-1}(xg_n(x))$ will be an arc joining $g_n^{-1}(x)$ to x. It will therefore contain a point y_n on S. Now g_n takes the pair (y_n, x) to a pair $(g_n(y_n), g_n(x))$ both elements of this pair being near x. The points y_n will have a limit y on S and there will have to be a g in G taking both y and x to x. This contradiction establishes the lemma.

LEMMA 6.3. *If* $g_n \to g$ *and* $g_n' \to g'$, *then* $g_n g_n' \to gg'$.

Let x be any point of E. Then $g_n'(x) \to g'(x)$. Hence $g_n[g_n'(x)] \to gg'(x)$ which we wanted to prove.

THEOREM 6.2. *Let* $(a_1, \cdots, a_k)$ *and* $(A_1, \cdots, A_k)$ *be any two sets of* k *points. Let* $(a_1^n, \cdots, a_k^n)$ *approach* $(a_1, \cdots, a_k)$ *and let* $(A_1^n, \cdots, A_k^n)$ *approach* $(A_1, \cdots, A_k)$. *If there is for each* n *an element* g_n *in* G *such that*

$$g_n(a_i^n) = A_i^n, \qquad i = 1, \cdots, k,$$

then there exists an element g *in* G *such that*

$$g(a_i) = A_i, \qquad i = 1, \cdots, k.$$

Let g_n' be an element of G such that $g_n'(a_i) = a_i^n$. Then $g_n g_n'(a_i) = A_i^n \to A_i$. Letting $i=1$ we see that $g_n g_n'(a_1) \to A_1$. Since a_1 certainly approaches a_1, Theorem 6.1 tells us that the sequence $g_n g_n'$ contains a subsequence converging to an element g^* in G. We assume that our original sequence above is taken as this convergent subsequence. An application of Theorem 6.1 to the sequence g_n' (remembering that $g_n'(a_1) = a_1^n \to a_1$) shows that for a subsequence of n's (which we now take for the whole sequence) $g_n' \to g'$ where g' is in G. We have now arrived at the following situation:

$$g_n g_n' \to g^*, \qquad g_n' \to g'.$$

By the preceding results $g_n'^{-1} g_n^{-1} \to g^{*-1}$. Hence

$$g_n'(g_n'^{-1}g_n^{-1}) = g_n^{-1} \to g'g^{*-1}, \qquad g_n \to g^*g'^{-1}.$$

Now for any i $(1 \leq i \leq k)$

$$A_i^n = g_n(a_i^n) \to g^*g'^{-1}(a_i).$$

But $A_i^n \to A_i$ and hence

$$g^*g'^{-1}(a_i) = A_i.$$

The element $g^*g'^{-1}$ therefore has the desired properties.

This theorem applied to the plane case (and with $k=3$) yields Hilbert's Axiom III. It is worth pointing out that we have actually proved a great deal more than Hilbert's axioms; we have also obtained many of the results of his paper. But as we are, at the moment, only interested in establishing that our weaker axioms suffice for the plane, we leave the plane case and turn our entire attention to the three dimensional case. From now on it is to be understood that we are dealing with this latter case.

13. By a *G-straight*, or more simply a *straight* or a *line* we shall mean a topological line which is the set of points left fixed by a circular subgroup of some G_x. Through every point of space there is clearly a large family of straights. Now let x and y be any two distinct points of the space. In the group G_x there is one and only one circular subgroup leaving y fixed. Let this group be called K_{xy}. It clearly leaves fixed a topological line, call it L_{xy}. We have therefore seen that there is at least one G-straight through every two distinct points of space. We know that L_{xy} is the set of *all* fixed points of K_{xy}. We have, therefore, the following theorem.

THEOREM 7. *Through each pair of distinct points of E there passes one and only one G-straight.*

13.1. It is worth noting that if an element g in G leaves fixed three points x, y, z not all on the same straight then g leaves all of E fixed. For since g leaves x fixed it is in G_x. Since it leaves y fixed it is in the circular subgroup K_{xy} of G_x; and since it leaves fixed z, a point not on the "axis" of K_{xy}, it must leave all of E fixed.

The symbol xy will be used to denote the closed portion of the line L_{xy} which is contained between the points x and y. This set of points will be called an *interval* or the *interval* xy, or a *segment*.

Let L_{xy} be a straight, left fixed by the circular group K_{xy}, and let g be any element of G. Then the set of points $g(L_{xy})$ is the set left fixed by $gK_{xy}g^{-1}$. In other words the image of a straight under any element of G is also a straight. It follows that the image of a segment is a segment.

Two configurations of the space E are said to be *congruent* if one of them is carried into the other by some element of G.

Any two straights are congruent and in fact any two marked straights are congruent. By a *marked straight* we mean a straight with some one of its points particularly "marked." It is furthermore true that any marked straight can be taken to any other so that a given direction on the one goes to a given direction on the other. These facts follow from the transitivity of G and the nature of the rotation group.

14. As we have said, a sphere is defined to be any two dimensional orbit of any group G_x. The point x is called the center of the sphere.

THEOREM 8. *A straight and a sphere can intersect in two points at most.*

Let x be the center of S the sphere and let L be the straight. Suppose a and b are two points of intersection of L and S. There exists an element of G_x which interchanges a and b. This element must carry L into itself since L is determined by any two of its points. Then there exists a non-trivial subgroup of G_x which leaves L invariant. This subgroup which will be denoted by Q is compact. Such a compact group acting on a line can contain only two distinct transformations, the identity and a reflection. Hence under this group the orbit of a consists of the two points a and b. It follows that there can be no other point c of L on S, for otherwise there would be an element of Q interchanging a and c and a would have at least three points in its orbit. This completes the proof.

THEOREM 9. *If p and q are inside a sphere S, then the segment pq is inside S.*

The straight L_{pq} is not compact in either direction. In going along L_{pq} from p to q and on we must meet S in some point. Similarly we must meet S in going from q to p and on out. The straight L_{pq} meets S, then, in two points neither of which is in the interval pq. Therefore no point of the interval can be outside S for this would imply that some point of the interval was on S and this would mean that L_{pq} had at least three distinct points on S.

15. THEOREM 10. *If $x_n \rightarrow x$ and $y_n \rightarrow y$ then $x_n y_n \rightarrow xy$.*

Let z_n, $n=1, 2, 3, \cdots$, be a point of $x_n y_n$. Any sphere surrounding xy surrounds almost all the z_n, so that for a proper subsequence of the n and a suitable point z, $z_n \rightarrow z$. We have to prove that z is on xy. There exist elements g_n such that $g_n x_n = x$, and $g_n y_n$ is on L_{xy} on the same side of x as the point y.

Now we know that there is a subsequence of the g_n and an element g such that $g_n \rightarrow g$. Further $gx = x$, $g_n y_n \rightarrow gy$, and $g_n z_n \rightarrow gz$. Now $g_n y_n$ is on L_{xy} so that gy must also lie on L_{xy}. On the other hand gy belongs to $G_x(y)$. Then it is clear that $gy = y$. Therefore g belongs to K_{xy}, g^{-1} belongs to K_{xy} and $z = g^{-1}(gz)$ is a point of L_{xy} since gz is a point of L_{xy}. It is now a trivial matter that z is on xy and that every point of xy is a limit point of some sequence z_n.

It should be remarked that under the same hypothesis the line $L_{x_n y_n}$ converges to the line L_{xy}. By this we mean that every sequence z_n of points from

$L_{x_ny_n}$ either has no limit point or every limit point which it has is on L_{xy}. Furthermore every point of L_{xy} is a limit point of such a sequence.

Let us now take a point z_n, $n=1, 2, 3, \cdots$, on $L_{x_ny_n}$ and assume that the sequence z_n converges to a point z. The intervals x_nz_n then converge to xz and y_nz_n converge to yz. It may be assumed that the points z_n are outside the interval x_ny_n say in the order $x_ny_nz_n$. The segment xz then contains xy and yz, so that xz is clearly part of the line L_{xy}.

The argument that every point of L_{xy} is such a limit is not difficult.

16. THEOREM 11. *Let x and y be any two points of the sphere S. Then the interval xy, except for x and y, is inside S.*

Let x_n and y_n be sequences of points inside S converging respectively to x and y. By a preceding theorem the intervals x_ny_n must be contained in the interior of S. The limit of these intervals will then be inside or on S. This limit is xy. This shows that xy is entirely contained in S and its interior. But L_{xy} can have no point besides x and y on S. The conclusion therefore follows.

17. A point y is said to be the midpoint of the segment or interval xz if it is on xz and if there is an element of G leaving y fixed and interchanging x and z. The point y is then the center of a sphere containing x and z as antipodal points.

THEOREM 12. *Every segment has a unique midpoint.*

We begin by showing the existence of the midpoint. Let y be a variable point of the segment xz. There is in G_y an element of order two which moves x to a unique point of L_{xz}. Let this unique point be denoted by $f(y)$. Now $f(x)=x$; and $f(z)$ lies on L_{xy} and has the order $xzf(z)$. Therefore if $f(y)$ is continuous it will assume for some y the value z. We have only to show therefore that $f(y)$ is continuous.

Let y_n approach y_0. Lemma 6 shows that $f(y_n)$ is a bounded set, and we may assume that $f(y_n)$ approaches a point w of the line L_{xz}. We wish to show that $w=f(y_0)$. Assume that this is not true and that w is distinct from $f(y_0)$. The pair (x, y_n) is carried by an element of the group to the pair $[f(y_n), y_n]$. There must be an element g of the group taking the pair (x, y_0) to the pair (w, y_0). There is also an element g' taking the pair (x, y_0) to the pair $(f(y_0), y_0)$. That is

$$g(y_0) = g'(y_0) = y_0$$

and

$$g(x) = w, \qquad g'(x) = f(y_0).$$

Hence $g'g^{-1}$ leaves y_0 fixed and takes w to $f(y_0)$. This is impossible because $g'g^{-1}$ is in the compact group G_{y_0} and the points w and $f(y_0)$ are both on the same side of y_0 on the line L_{xz}. Hence $f(y)$ is continuous and the midpoint of xz exists.

Assume now that there are two midpoints y and y' of the segment xz. Each of them gives rise to a reflection interchanging x and z and leaving itself fixed. The product of these two reflections of L_{xz} is a transformation of L_{xz} leaving x and z fixed and moving other points on the line. This is impossible.

18. The next geometric concept to be defined will be the *projection* of a point z on a line L. If z is on L this projection is defined to be z itself. If z is not on L let g be an element of order two in K_L which is the circular group leaving every point of L fixed. Let $z'=g(z)$; evidently $g(z')=z$. The line $L_{zz'}$ and the segment zz' are invariant under g, so that g is a reflection of the line and segment with a fixed point p. Since this point p is fixed under an element of K not the identity, it is fixed under all of K and is on L. The point p is now defined to be the projection of z on L.

THEOREM 13. *The projection of z on L is a continuous function of z.*

Suppose z_n converges to z. Then $g(z_n)=z_n'$ converges to a point z', g being the element of order two in the group leaving L fixed. The segments z_nz_n' converge to the segment zz'. The points p_n have some limit point on zz'. But a limit point of the p_n's must be fixed under g and can only be the point p.

19. The space E is given to us as a metric space. Our purpose now is to introduce a new metric equivalent to the old which will be invariant under G.

Let L be any straight in E and let G^* be the set of elements of G which transform L into itself while preserving direction. Any element of G^* which leaves a point of L fixed leaves every point of L fixed. Let g_n be a sequence of elements of G^* such that for some point a in L the sequence $g_n(a)$ approaches a. Then for any b in L the sequence $g_n(b)$ approaches b. It follows from the fact mentioned here that if two homeomorphisms of G^* act approximately the same way on a single point of space then they act approximately the same way over any bounded part of the line.

It will be useful to note also that if ab is an interval of L and g takes L into itself with direction reversed and if $g(b)=a$ it follows that $g(a)=b$. With the aid of G^* we shall now see how L becomes the carrying space of a topological group.

19.1. Let o be a point of L fixed but arbitrary. Let a and b be any two points of L. Let f be an element of G^* which moves o to a, and let g be an element of G^* which moves o to b. Then by definition $a\cdot b=fg(o)$. This definition of multiplication on L is associative. It has an inverse, and hence all the group axioms are satisfied. If $f(o)=a$, the inverse of a is $f^{-1}(o)$. Another way to obtain a^{-1} is to define it as the position to which a goes by an element which reverses direction on L while leaving o fixed. The latter definition shows that the operator "inverse" is continuous.

Now assume that a_n approaches a and b_n approaches b. Let $f_n(o)=a_n$,

$g_n(o)=b_n$, $f(o)=a$, and $g(o)=b$. By the remarks in §19 f_n is near f for large n and any bounded portion of the line. A similar statement may be made about g_n and g. Therefore $f_n g_n(o)$ is near $fg(o)$. This shows that the group multiplication $a\cdot b$ is simultaneously continuous in a and b.

Thus with multiplication as above defined L becomes the carrier of a topological group. But it is known that such a group must be bicontinuously isomorphic to the additive group of real numbers. The line L may now be metrized with the metric of the real numbers carried by it in this way. That is, if x and y are any two points of L, then $d^*(x, y)$, the new distance, is defined to be the absolute value of the difference of the real numbers corresponding to x and y.

Now suppose that the segment xy is translated to $x'y'$ by an element g of G^*. Thus:

$$g(x) = x', \qquad g(y) = y'.$$

If $g(o)=z$, then

$$z\cdot x = x', \qquad z\cdot y = y'.$$

Therefore $x\cdot y$ is translated to $x'\cdot y'$ by an operation of the topological group defined above and therefore $d^*(x, y)=d^*(x', y')$.

Any element of G which reverses sense on L also preserves the new distance. This is because there is one sense reversing transformation which merely changes the ends of a given interval. This transformation leaves the length of the interval invariant. Any other sense reversing transformation is the product of this one and an element of G^*, from which our statement now follows.

19.2. We may now define the new distance for any two points x and y of space. There is some element in G which carries x and y to two points x' and y' of L. The distance $d^*(x, y)$ is defined as equal to $d^*(x', y')$. By its very definition this new distance extended as it now is to all of E is invariant under G. In fact any two pairs of points are congruent if and only if they have the same distance d^*.

19.3. To show that the new distance d^* is equivalent to the old it is only necessary to show that it is a continuous function.

THEOREM 14. *The distance $d^*(x, y)$ is a continuous function of x and y.*

Let x_n's converge to x and y_n's to y. We saw in the proof of Theorem 10 that G contains an element which takes x_n to x and y_n to the line L_{xy}, near y. But on L and hence on L_{xy}, d^* is a continuous function. This shows that d^* of the transformed pair, which is the same as d^* for the original pair (x_n, y_n), is near $d^*(x, y)$.

19.4. THEOREM 15. *The distance $d^*(x, y)$ satisfies the triangle axiom.*

Let A, B, and C be any three points of E. We wish to prove that

$$d^*(A, C) + d^*(C, B) \geqq d^*(A, B).$$

If C is on the line L_{AB} we know this. When C is between A and B the equality holds, and this will be the only case of equality.

Suppose now that C is not on L_{AB} and let C' be the projection of C on the line. To prove the desired inequality it will suffice to show that $d^*(A, C) > d^*(A, C')$ and $d^*(C, B) > d^*(C', B)$. This follows from the following lemma.

Lemma 7. *Let x, y, and z, not on a line, be such that z is the projection of y on L_{xz}. Then $d^*(x, y) > d^*(x, z)$.*

Let S be the sphere about x which goes through y, that is, the sphere with center x and radius $d^*(x, y)$. Let g be the element of order two in K_{xz} and let $y' = g(y)$. The segment yy' has z as midpoint, so that certainly z is on yy'. The point y' is on S because xy and xy' are congruent under g. If now $d^*(x, z) = d^*(x, y)$, there would have to be a point z' on the segment xz such that $d^*(x, z') = d^*(x, y)$. Since y and y' are on the sphere, z is inside it. Of course x is inside the sphere and hence z' is also inside it, contrary to the preceding equality.

It should be noted that all points of yy' except y and y' are inside S so that for any point w of yy' distinct from y and y', $d^*(x, w) < d^*(x, y)$.

It is clear that we have established the metric characterization of a straight line: three points are on a straight line if and only if their distances in proper and unique order satisfy the triangle equality. From this point on, the distance d^*, now called d, will be the only distance used.

20. Let L and L' be two lines having in common the single point p and such that there exists an element of order two in K_L under which the line L' is invariant. In this case L' is said to be *orthogonal* or *perpendicular* to L. If x is a point on L', its projection on L is the point p.

Consider the rotation group G_p and the coordinate system that goes with it which makes G_p the three-space rotation group. Since there is an element of order two, that is, a half-rotation in G_p which leaves L fixed and L' invariant, there is also a half-rotation in G_p which leaves L' fixed and L invariant. In other words L is also orthogonal to L' and the relation of orthogonality is symmetric. It can be seen also that orthogonality is a group invariant.

20.1. The transitivity of G and the nature of the rotation group enables us to state the following two theorems.

Theorem 16. *Let L and N be two lines both orthogonal to a line M at a point p. There is then an element of K_M which carries L to N.*

This makes it clear incidentally that the locus of points on lines orthogonal to L at p is a topological plane.

THEOREM 17. *Let L and L' be orthogonal at p and let M and M' be orthogonal at q. There is then an element g in G such that $g(p)=q$, $g(L)=M$ and $g(L')=M'$. Furthermore this element may be so chosen as to take any desired directions on L and M to any desired directions on L' and M'.*

20.2. Let L and L' be a pair of orthogonal lines intersecting in a point p. Let x and y be two points of L' on opposite sides of p and at equal distances from p. A half-rotation about L must then interchange x and y. If q is any point of L, the half-rotation carries the segment qx to qy, so that q is equidistant from x and y. We may express this result by saying that if two lines are orthogonal any point on one of them is equidistant from any pair of symmetrically placed points on the other.

20.3. A triangle is defined in the natural way as the system of three segments joining pairs of a set of three points. Other simple geometric concepts will sometimes be used without definition when the definition is perfectly straightforward.

As a sort of converse to the result of the preceding section the following theorem is given at this point.

THEOREM 18. *The altitude of an isosceles triangle bisects the base.*

Let q, x, and y be three points such that qx and qy are congruent. Let p denote the projection of q on L_{xy}. Then L_{pq} is orthogonal to L_{xy}. Consider the half rotation which leaves L_{pq} fixed and L_{xy} invariant. This carries x and y to x' and y', all four points being on the same line. All four segments qx, qy, qx', and qy' are equal and the line L_{xy} must meet the sphere $G_q(x)$ in four points, x, y, x', and y'. At most two of these are distinct and we know that x and y are distinct. This makes it clear that y' is x and x' is y, so that p is the midpoint of the segment xy.

21. Let L denote a line and p a point on it. Let π denote the set of all points of space whose projection on L is the point p. Such a set, by definition, is a plane of our geometry. It is clear that any line M orthogonal to L at the point p belongs to π, and that every point of π is on one such line. Now let x and x' be a pair of points symmetrically situated about p on the line L. Then all points of π are equidistant from x and x': conversely any point equidistant from these must lie on π. Our planes may therefore be characterized as the locus of points equidistant from some pair of points. If we consider the circle group which leaves fixed the line $L=L_{xx'}$, we see that every line M, as above, is generated by this group from any arbitrary one. It follows at once that our planes have the topological structure which they should.

21.1. To show that our planes are linear sets we shall borrow from Kerékjártó the notion of introducing a simple antipodal transformation α, *not* an element of G, defined on the whole space as follows. Under α the point p is fixed: a point q goes to that point q' *on the line pq* which is symmetrically disposed about p. It is obvious that the straight lines through p, and only these, are invariant under α.

Our metric is invariant under this transformation α. To see this, following Kerékjártó, we need merely show that, for an arbitrary *pair of points*, α coincides with an appropriate element of G. To this end, let q and s be two points, distinct from p, and let L' denote a line orthogonal to *pq and ps*. Let g denote a half-turn about L'. It is clear that q and s have the same images under this half-turn as under the antipodal transformation α. It now follows immediately from the invariance of our metric that α *carries straight lines of space to straight lines*, for these as we have seen are metrically characterized.

21.2. Consider now the line L and the plane π orthogonal to it at the point p. Perform upon space the antipodal transformation α followed by a half-turn about L. It is clear that the points of π are fixed points under this product-transformation. But it is important to observe that they are the only fixed points: this follows from the fact that all lines through p are invariant under α and only those *remain* invariant under the half-turn which are orthogonal to L at p. Now take two points, q and s, in π. The line through these must be invariant because our composite transformation carries it to a line through q and s, since these are fixed points. This transformation, moreover, preserves distances. Since it has a pair of fixed points it must leave all of the line fixed. Therefore the line through q and s must lie in π, the locus of fixed points. Then we have shown that with every pair of its points our plane contains the line determined by these points.

22. As we know, the plane π is invariant under the circle group K_L. We want to show next that associated with every point q of π there is a similar circle group leaving π invariant. Such a step will enable us to see that there is nothing special about the point p and to conclude that π is either a euclidean or hyperbolic plane under the subgroup G^π of G which leaves π invariant. This group is transitive on π.

22.1. Let us introduce the notion of the projection of a point x on the plane π. This is defined as that point x' of π which is nearest to x. In order to see that x' is determined, let z denote a point of π and let S denote a sphere with center at x and radius xz. The solid sphere intersects π in a compact set. For any point of π not in this intersection the distance to x must exceed xz. On the compact set there certainly is one "nearest" point, but conceivably more than one. Now there can be at most one such nearest point. For suppose x' and x'' are two points of π at the same distance from x. Then the midpoint of $x'x''$ is in π and is nearer to x than x' and x'' are. It is clear from the continuity of distance and the uniqueness of projection that this projection operation is continuous. We shall use this continuity in the following theorem.

THEOREM 19. *Every point q of π is the projection of at least one point q' not on π.*

Consider a sphere about P large enough to have the point q inside of it. This sphere meets π in a circle, call it C. Let S denote one of the hemispheres associated with C. Let S^* denote the set of projections of S. Since C can be deformed on S to a point of S, it can be deformed on S^* to a point of S^*. During this deformation it must meet q since q is in the domain bounded by C. This means that q is a point of S^* as was to be shown.

22.2. Let q' be a point not on π and let q, distinct from p, be its protection on π. Let π' denote the plane of lines orthogonal to qq' at q. This plane contains all straight lines of which it contains a pair of points, by 21.2.

THEOREM 20. *The planes π and π' are identical.*

It will first be shown that π' includes π. Consider the line L_{pq}. This line is in both planes. The coordinate system around P shows us that there is one and only one line in π which goes through p and is orthogonal to L_{pq}. Let L' denote any line, distinct from this one, through p and in π. Let s denote the projection of q on the line L'. The point s is distinct from p by our choice of L'. Since s and q are points of π, L_{sq} belongs to π. Since it is a line of π which goes through q it is also in π'. Therefore s and p are both points of π' and L_{ps} belongs to π'. The set of points on such lines is dense in π, and the closure of this set, which is π, must also belong to π'.

We will now show that π contains π'. The plane π has the property that each of its points is interior to a two-cell, and it must therefore be an open set in π'. On the other hand it is a closed subset of space and is therefore closed in π'. Then it must coincide with π'.

23. Let (π, p) denote a *marked plane*, that is to say a plane π with some one of its points p particularly specified.

THEOREM 21. *Any two marked planes are congruent.*

Let (π, p) and (σ, s) be the two marked planes. Let L denote a line orthogonal to π at p. Such a line shall be by definition a line through p orthogonal to every line of π through L. This orthogonal line always exists and is unique, for it is the locus of points p' which project on p when they are projected on π. This we will see as follows. Let L^* be the totality of points projecting on p. As we know L^* must contain at least one point p' distinct from p. Hence from previous considerations L^* must be the line $L_{pp'}$.

Now let L' denote a line orthogonal to σ at s. The marked line (L, p) may be carried to the marked line (L', s). Since these lines completely determine π and σ, (π, p) must go into (σ, s) by the element which takes (L, p) to (L', s). If we carry the line L' into itself by any half turn about the point s, the plane σ must go into itself with orientation reversed. The marked planes are therefore congruent with a matching of any orientations that we choose on them.

24. The geometric concepts have now been analyzed sufficiently for us to

be able to see that they satisfy the axioms usually given for a geometry, with the exception of the parallel axiom. Axioms I, II, III, and V as given by Hilbert [4] for example are all satisfied.

It is clear that there are two possible geometries satisfying our axioms for the space case, the euclidean and the hyperbolic. We now sketch rapidly one method for seeing that there are not more than two.

Let π be any plane in E. The subgroup of G which takes π into itself and preserves orientation on π can be seen to satisfy the axioms of Hilbert's paper [5]. The plane is therefore either euclidean or hyperbolic.

Since all planes are congruent to any given plane, we see that either every plane is hyperbolic or every plane is euclidean. We wish to show that the geometry induced by G is either euclidean or hyperbolic according to the character of the planes.

Let (E, G) and (E', G') be two systems satisfying all the axioms for the space case and assume that in the two systems planes are of the same character.

Let (π, p) be a marked plane in E and let (π', p') be a marked plane in E'. Let H denote a congruence correspondence between these two planes. Let L and L' be the unique lines of E and E' which are orthogonal to π and π' at p and p'. The unit of length gives us a unique correspondence between L and L', the only choice, and it is an arbitrary one, being which half of L is mapped on which of L'. Assume that this choice has been made so that we have really chosen an upper and a lower half for E and also for E'.

We can now choose coordinates in E and E' and extend H by letting points with the same coordinates correspond. The correspondence H as thus extended is isometric. It preserves segments, orthogonality, lines, planes, and in fact all geometric concepts. The function H also associates with every g in G an element g' in G' and we are therefore led to the conclusion that (E, G) and (E', G') are equivalent provided the planar character of the two systems is the same.

APPENDIX

For the space case an alternative set of axioms might be chosen as follows. Let (E_3, G) be a system consisting of a set G of sense preserving homeomorphisms of E_3. Let the following axioms be satisfied.

2.1″. *The same as* 2.1′.

2.2″. *There is a point p such that G_p satisfies the conditions* (a), (b), and (c):

(a) *G_p is a proper subgroup of G.*

(b) *For each x distinct from p, $G_p(x)$ contains at least three points.*

(c) *For a sequence of points p_n approaching p, $G_p(p_n)$ is at least one dimensional.*

2.3″. *The same as* 2.3′.

We will show that these axioms imply 2.2′ from which it follows that they suffice for the foundation of space geometry.

Quite as in the paper we can arrive at the situation of §5. We have then an open set R_1 with a closure $\overline{R}_1$, and $\overline{H}_1$ is a compact effective transformation group of $\overline{R}_1$. The orbits of $\overline{H}_1$ are the same as those of G_p.

Let H_1^* be the component of the identity of $\overline{H}_1$. The set $H_1^*(x)$ has the same dimension as $\overline{H}_1(x)$ and $\overline{H}_1$ must have orbits of dimension at least one in $\overline{R}_1$.

Assume now that H_1^* is one dimensional. Then H_1^* is the circle group. As H_1^* leaves fixed an "axis" of points of $\overline{R}_1$, not every x distinct from p has 3 points in its orbit under H_1^*. Therefore H_1^* does not exhaust $\overline{H}_1$. But H_1^* divides $\overline{R}_1$ locally (near p) into a decomposition space which is essentially a half plane. $\overline{H}_1-H_1^*$ must act on this half plane and the only compact group which can act on a half plane is a group of order two which reflects its edge. Hence in any case points on the "axis" of H_1^* will have orbits of at most two points under $\overline{H}_1$. This shows that $\overline{H}_1$ cannot be one dimensional. But since it is now seen to be of dimension greater than one it must also have orbits of dimension greater than one by arguments in our earlier papers. This concludes the reduction of the present system of axioms to those of this paper.

It should be remarked here that in the presence of condition (b) above it is altogether likely that condition (c) can be relaxed perhaps merely to assert, with Hilbert, that $G_p(p_n)$ is infinite. This appears to have all of the difficulty which attends the problem of showing, if it is true, that a zero dimensional topological transformation group of three-space is necessarily finite.

We might mention in conclusion that it seems to us of some interest to determine the three-space geometries through appropriate reflection groups, along the lines on which this was done for the plane by Cairns. While it is clear that suitable conditions on "reflections" of three-space could be made to yield the axioms of this paper, the characterization of the fixed points of reflection of three-space by P. Smith might lead to an interesting approach.

Added in proof: Kerékjártó has informed us that he has published a further paper on this same subject in the Proceedings of the Hungarian Academy of Sciences (1928) (in Hungarian). He is about to publish another paper on this subject in the Acta Mathematica.

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TWO-DIMENSIONAL SPACES IN WHICH THERE EXIST CONTIGUOUS POINTS

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Reprinted from the
Transactions of the American Mathematical Society
Vol. 44, No. 2, pp. 250-276
September, 1938

TWO-DIMENSIONAL SPACES IN WHICH THERE EXIST CONTIGUOUS POINTS*

BY

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In a recent number of The Rice Institute Pamphlet† R. L. Moore has formulated a set of axioms in terms of the undefined notions "point," "region," and "contiguous to." These axioms‡ (Axioms A, B, C, 0, 1, and 2 of this paper) serve as the basis for the proofs of a considerable number of theorems of ordinary point set theory, including a large proportion of the theorems of the first two chapters of Moore's book.§ Nevertheless, there exist spaces satisfying these axioms in which an arc may contain only a finite number of points and in which a region may consist of a finite number of points.

In the present paper a study is made of spaces which satisfy the above mentioned axioms and some additional axioms which restrict the spaces to being, in a certain sense, two dimensional. The ordinary euclidean plane is a space which satisfies all the axioms.

I wish to acknowledge my indebtedness to Professor R. L. Moore, and to thank him for suggesting the problem and for many helpful criticisms in the course of its development.

For definitions of terms used but not defined here the reader may refer to S.C.P.

DEFINITIONS. *A simple closed curve is a compact continuum, containing at least two distinct non-contiguous points, which is disconnected by the omission of any two of its non-contiguous points. A triune is a set of three distinct points such that each of them is contiguous to each of the others.*

AXIOM A. *No point is contiguous to itself.*

AXIOM B. *If the point A is contiguous to the point B, then B is contiguous to A.*

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* Presented to the Society, October 28, 1933; received by the editors August 18, 1937.

† Vol. 23 (1936), no. 1. In the present treatment the abbreviation S.C.P. will be used to designate part 1 of this paper.

‡ It being understood that in the statement of Axiom 2 of S.C.P. the word "non-degenerate" is to be omitted. It is clear from the context that the retention of this word was not intended.

§ *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. In the present treatment the abbreviation P.S.T. will be used to designate this book.

Axiom C. *If M is a closed point set and every point of the set H is contiguous to some point of M, then no point of $S-M$ is a limit point of H.*

Axiom 0. *Every region is a point set.*

Axiom 1. *There exists a sequence $G_1, G_2, G_3, \cdots$ such that (1) for each n, G_n is a collection of regions covering S, (2) for each n, G_{n+1} is a subcollection of G_n, (3) if R is any region whatsoever, and if X is a point of R and Y is a point of R either identical with X or not, then there exists a natural number m such that if g is any region belonging to the collection G_m and containing X, then $\bar{g}$ is a subset of $(R-Y)+X$, (4) if $M_1, M_2, M_3, \cdots$ is a sequence of closed point sets such that M_n contains M_{n+1} for each n and there exists a region g_n of the collection G_n such that M_n is a subset of $\bar{g}_n$ for each n, then there is at least one point common to all the point sets of the sequence $M_1, M_2, M_3, \cdots$.*

Axiom 2. *If P is a point of a region R, there exists a connected domain containing P and lying in R.*

Axiom 3. *If J is a simple closed curve or triune, then $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.*

In this paper I deal very frequently with complementary domains of simple closed curves and triunes. If J is a simple closed curve or triune and ω is a point of $S-J$, then by *the interior of J with respect to ω*, is meant that complementary domain of J which does not contain ω. Similarly, that complementary domain of J which contains ω is called *the exterior of J with respect to ω*. In case no ambiguity arises, the terms *interior* and *exterior* of J will be used without making specific reference to a point ω.

Theorem 1. *Let J_1 and J_2 denote two point sets each of which is either a simple closed curve or a triune. Let I_1 and I_2 denote the interiors of J_1 and J_2, respectively, and suppose J_2 is a subset of J_1+I_1. Then I_2 is a subset of I_1.*

Theorem 2. *If J is a simple closed curve, and AB separate C and D on J, and AXB is an arc such that the segment AXB is a subset of I, the interior of J, and no point of the segment AXB is constiguous to any point of $J-(A+B)$, then (1) I_1, the interior of $AXBCA$, is a subset of I, (2) the segment ADB is a subset of the exterior of $AXBCA$, and (3) I_1 has no point in common with I_2, the interior of $AXBDA$.*

Theorem 3. *Under the hypothesis of Theorem 2, $I=I_1+I_2+$segment AXB.*

Proof. Suppose $I(I_1+I_2+\text{segment } AXB)=M$, where M is a non-vacuous point set. Let P be a point of M. By Theorem 38 of S.C.P. there exists an arc PX from P to X lying in I. The arc PX contains an arc PX' such that

$PX'-X'$ is a subset of M, and X' is a point of the segment AXB. Similarly, there exists an arc $P\omega$ lying in E_2, the exterior of $AXBDA$. The arc $P\omega$ contains an arc PC', such that $PC'-C'$ is a subset of M, and C' is a point of the segment ACB. Let T denote the first point in the order from X' to P which $X'P$ has in common with PC'. If no point of the interval $X'T$ of $X'P$, except possibly T, is contiguous to any point of $(TC'-T)$, where TC' denotes the interval of PC' from T to C', then $X'T+TC'$ is an arc from X' to C'. If there exist points of $X'T-T$ which are contiguous to points of $TC'-T$, there must be a first such point in the order from X' to T. For otherwise, there would be infinitely many such points, and the set of all such points would have a limit point in $X'T-T$. But by Axiom C, every limit point of such a set must belong to TC', and a contradiction is reached. Let W denote the first point of $X'T-T$ which is contiguous to a point of $TC'-T$. By Axiom C, W is contiguous to only a finite number of points of $TC'-T$. Let V denote the last point in the order from T to C' which is contiguous to W. Let $X'W$ denote the interval of $X'T$ from X' to W or the point X' according as W is not or is identical with X'. Let VC' denote the interval of TC' from V to C' or the point C' according as V is not or is identical with C'. The point set $X'W+VC'$ is an arc containing at least three points. Thus, in any case, the point set $PX'+PC'$ contains an arc $X'P'C'$, such that the segment $X'P'C'$ contains at least one point and is a subset of M. Similarly, we may show the existence of an arc $X''P''C''$ such that (1) the segment $X''P''C''$ contains at least one point and is a subset of I_1 and (2) the points X'' and C'' are points of the segments AXB and ACB, respectively. Let $X'X''$ denote the point X' or the arc of the segment AXB from X' to X'', according as X'' is or is not identical with X'. Also let $C'C''$ denote the point C' or the arc of the segment ACB from C' to C'', according as C'' is or is not identical with C'. By means of repeated applications of Axiom C it may be shown that the point set $X'P'C'+C'C''+X''P''C''+X'X''$ contains a simple closed curve J', which contains at least one point of each of the segments $X'P'C'$ and $X''P''C''$. Let I' denote the interior of J'. Now I' is a subset of I by Theorem 1. Thus the segment ADB is a subset of E', the exterior of J'. Since the connected set I_2 plus the segment ADB contains no point of J' but does contain a point of E', it follows that I_2 plus the segment ADB is a subset of E'. Furthermore I' cannot contain a point of the segment AXB. For, suppose I' contains the point F of the segment AXB. The connected set $F+I_2$ contains no point of J' but contains points of both complementary domains of J'. We conclude from this contradiction that the segment AXB is a subset of $J'+E'$. Now I' cannot be a subset of M since there exists a point of $J'\cdot I_1$ which is either a limit point of I' or contiguous to a point of I', and M and I_1 are two mutually separated domains. Also, I' cannot be a subset of I_1 since there exists a point of $J' \cdot M$ which is either a limit point of I' or contiguous to a point of I', and M and I_1 are two mutually separated domains. Also I' cannot be a subset of $M+I_1$ and contain points of both M and I_1 since I' would thus be the sum of two mutually separated sets, contrary to Axiom 3. Thus in any case we reached a contradiction, and the theorem is established.

THEOREM 4. *If the points A and B separate the points C and D on the simple closed curve J, and if the segments AXB and CYD are both subsets of I, the interior of J, then these segments have at least one point in common.*

Proof. Suppose the segments AXB and CYD have no point in common. There exists an arc $A'\omega$ such that $A'\omega - A'$ is a subset of E, the exterior of J, and such that A' is a point of the segment CAD of J. Similarly, there exists an arc $B'\omega$ such that $B'\omega - B'$ is a subset of E and B' is a point of the segment CBD. The point set $A'\omega + B'\omega$ contains an arc $A'X'B'$ such that the segment $A'X'B'$ is a subset of E. Let AA' denote the point A or the arc of the segment CAD from A to A' according as A' is or is not A, and let BB' denote the point B or the arc of the segment CBD from B to B' according as B' is or is not B. It may be shown that there exists a simple closed curve J^* satisfying the following conditions: (1) J^* is a subset of $AA'+BB'+AXB+A'X'B'$, (2) J^* contains at least one point of each of the segments AXB and $A'X'B'$, and (3) $J^* \cdot J$ is the sum of two mutually separated continua which separate C and D on J; therefore $J - J^* \cdot J = g_1 + g_2$, where g_1 and g_2 are mutually separated segments containing C and D, respectively. Since J^* contains no point of the arc CYD, it follows that the connected point set $CYD + g_1 + g_2$ is a subset of I^*, a complementary domain of J^*. Thus J is a subset of $I^* + J^*$. Hence, by Theorem 1, either I or E is a subset of I^*. But each of the sets I and E contains a point of J^*, and J^* contains no point of I^*. Thus we reach a contradiction, and the theorem is established.

THEOREM 5. *If J is a simple closed curve or triune, then I, the interior of J, contains infinitely many points.*

Proof. Suppose I contains exactly n points, where n denotes a natural number. Thus every point of J is contiguous to some point of I. If there are infinitely many points of J, the closed point set I contains a limit point of J by Axiom C. But this is impossible since J is closed. Hence J contains only a finite number of points. Let A and B be two contiguous points of J, and let X_0 be a point of $J-(A+B)$. Let X be a point of I which is contiguous to A and Y a point of I contiguous to B. Let XY denote the point X or an arc from X

to Y lying in I, according as Y is or is not identical with X. The point set $XY+A+B$ contains a simple closed curve or triune J_1 which is a subset of $J+I$ and contains A, B, and a point X_1 of I but is such that X_0 is in the exterior of J_1. Similarly, there exists a simple closed curve or triune J_2 which is a subset of J_1 plus its interior and contains A, B, and a point X_2 of the interior of J_1 but is such that X_1 is in the exterior of J_2. By continuing the indicated process $n+1$ times we reach a contradiction, since $X_1, X_2, X_3, \cdots, X_{n-1}$ are distinct points of I. Hence I must contain infinitely many points.

THEOREM 6. *Let J denote a simple closed curve whose interior I contains a point P which is contiguous to at least three distinct points of J. Let $P_1, P_2, \cdots, P_n$, $(n \geqq 3)$, be points of J (in the order indicated if $n>3$), and let $\beta = P_1+P_2+\cdots+P_n$. Suppose β is the set of all points of J that are contiguous to P. Let P_kP_{k+1}, $(k=1, 2, \cdots, n)$, denote that arc of J which contains only the points P_k and P_{k+1} of the set β, and let P_{n+1} denote P_1. Let I_k, $(k=1, 2, \cdots, n)$, denote the interior of the triune or simple closed curve $P+P_kP_{k+1}$. Then $I = P+\sum_{k=1}^{n} I_k$.*

Proof. By Theorem 1, I_k, $(k=1, 2, \cdots, n)$, is a subset of I. Also, if $k \neq j$, then I_k and I_j are mutually exclusive. Suppose $I-(P+\sum_{k=1}^{n} I_k) = M$, where M is a non-vacuous point set, and let M_1 denote a component of M. Now M is a domain; hence M_1 is a domain. Let α denote the set of all points of J, each of which is either a limit point of M_1 or contiguous to a point of M_1. There exists at least one point of α. For let Q_1 denote any point of M_1, and let $Q_1\omega$ denote an arc from Q_1 to ω lying in the exterior of the simple closed curve or triune $P+P_1P_2$. Let Q_1' denote the first point that $Q_1\omega$ has, in order Q_1 to ω, in common with J. Now Q_1' is obviously a point of α, since Q_1' is the first point of $Q_1\omega$, in order Q_1 to ω, which does not belong to M_1.

Suppose there exist two points X and Y of α, which do not lie together on one of the arcs $P_1P_2, P_2P_3, \cdots, P_nP_1$. Since X and Y are non-contiguous points of J, it follows that J is the sum of two arcs from X to Y having nothing in common except their end points such that the corresponding segments of these arcs are mutually separated. There exists a point P_r of β on one of these segments and a point P_s of β on the other. Since P_r and P_s are both points of β, P_r+P+P_s is an arc lying in $I+J-(M+X+Y)$. There exists an arc P_rQP_s which is a subset of $J+E-(X+Y)$, where E denotes the exterior of J, and is such that $P_rQP_s \cdot J$ is the sum of two mutually separated connected point sets M_r and M which contain P_r and P_s, respectively, but no other points of β, and which separate X and Y on J. The point set $P+P_rQP_s$ is a simple closed curve J' which contains P and is such that $J' \cdot J$ is the sum of two mutually separated connected point sets which separate X and Y on J.

Thus $J-J'\cdot J$ is the sum of two mutually separated segments g_X and g_Y which contain X and Y, respectively. Now J' separates X from Y, for otherwise g_X and g_Y would lie together in I', a complementary domain of J'; therefore by Theorem 1, either I or E would be a subset of I', which is contrary to the fact that both I and E contain points of J'. Since X and Y are boundary points of M_1 lying in different complementary domains of J', it follows that there are points of M_1 lying in different complementary domains of J'. This is impossible since M_1 contains no point of J'. From this contradiction we conclude that if X and Y are any two distinct points of α, then there exists an arc of the set P_1P_2, P_2P_3, $\cdots$, P_nP_1 which contains both X and Y.

Thus, if $J-\beta$ contains a point X of α, then the arc of the set P_1P_2, P_2P_3, $\cdots$, P_nP_1 which contains X must contain all of α. Now if α is not a subset of one of the arcs of the set P_1P_2, P_2P_3, $\cdots$, P_nP_1, it follows that $J-\beta$ contains no point of α. Furthermore, it follows that $n=3$ and $\alpha=\beta=P_1+P_2+P_3$. In this case, since J is not a triune, one of the segments P_1P_2, P_2P_3, P_3P_1 exists. Suppose the segment P_3P_1 is non-vacuous. Using methods similar to those used in the early part of the proof to get J', we obtain a simple closed curve J^* satisfying the following conditions: (1) J^* is a subset of $P+J+I_3+E$, (2) J^* contains P, P_2, a point of the segment P_3P_1, a point of E, and a point of I_3, and (3) $J^*\cdot J$ is the sum of two mutually separated connected point sets which separate P_1 and P_3 on J, and $J-J^*\cdot J$ is therefore the sum of two mutually separated segments, one containing P_1 and the other containing P_3. The argument used above to show that J' separates X and Y may be applied here to show that J^* separates P_1 and P_3. Hence J^* separates two points P_1' and P_3' of M_1, since P_1 and P_3 are points of α. But this is impossible since M_1 is connected and contains no point of J^*. Thus we reach a contradiction, and we conclude that there exists an arc P_iP_{i+1} of the set of arcs P_1P_2, P_2P_3, $\cdots$, P_nP_1 which contains α.

Let E_i denote the exterior of the simple closed curve or triune $(P+P_iP_{i+1})$. The domain E_i contains M_1, and we write $E_i=M_1+(E_i-M_1)$. The boundary of M_1 is a subset of $(P+P_iP_{i+1})$. Hence M_1 and E_i-M_1 are mutually separated, contrary to Axiom 3. Thus we reach a contradiction, and the theorem is established.

THEOREM 7. *Let P_1, P_2, $\cdots$, P_n, $(n\geqq 3)$, be points of the simple closed curve J (in the order indicated if $n>3$). Let A_1, A_2, $\cdots$, A_k, $(k\geqq 2)$, be points of the arc A_1A_k (in the order indicated if $k>2$) where A_1A_k is a subset of I, the interior of J. Let $\beta=P_1+P_2+\cdots+P_n$, and let $\gamma=A_1+A_2+\cdots+A_k$. Suppose β is the set of all points of J each of which is contiguous to at least one point of A_1A_k, and suppose γ is the set of all points of A_1A_k each of which is contiguous to at least one point of β. Suppose furthermore that no point of β is contiguous to both*

A_1 and A_k, but that P_1 is contiguous to A_1 and to no other point of γ and P_n is contiguous to A_k and to no other point of γ. It follows that if $j<j'$, and if P_j is contiguous to A_i and $P_{j'}$ is contiguous to $A_{i'}$, then $i \leqq i'$. Let P_jP_{j+1}, $(j=1, 2, \cdots, n)$, denote that arc of J from P_j to P_{j+1} which contains no other point of β, where P_{n+1} denotes P_1. Let A_iA_{i+1}, $(i=1, 2, \cdots, k-1)$, denote the arc of A_1A_k from A_i to A_{i+1}. Let I_j, $(j=1, 2, \cdots, n-1)$, denote the interior of C_j, where C_j denotes the triune or simple closed curve $(P_jP_{j+1}+A_i)$ in case P_j and P_{j+1} are both contiguous to A_i, or C_j denotes the simple closed curve $(P_jP_{j+1}+A_iA_{i+1})$ in case no point of γ is contiguous to both P_j and P_{j+1}, but where P_j is contiguous to A_i and P_{j+1} is contiguous to A_{i+1}. Let I_n denote the interior of the simple closed curve $(P_nP_1+A_1A_k)$. If P_j is contiguous to each of the points $A_{i_j}, A_{i_j+1}, \cdots, A_{i_j+k_j}$, let $I_{j1}, I_{j2}, \cdots, I_{jk_j}$ denote the interiors of $(P_j+A_{i_j}A_{i_j+1})$, $(P_j+A_{i_j+1}A_{i_j+2})$, $\cdots$, and $(P_j+A_{i_j+k_j-1}A_{i_j+k_j})$, respectively. Then

$$I = A_1A_k + \sum_{j=1}^{n} I_j + \sum_{j=1}^{n}\sum_{t=1}^{k_j} I_{jt},$$

where I_{jt} is a null set if P_j is contiguous to only one point of γ.

The theorem is illustrated by Fig. 1.

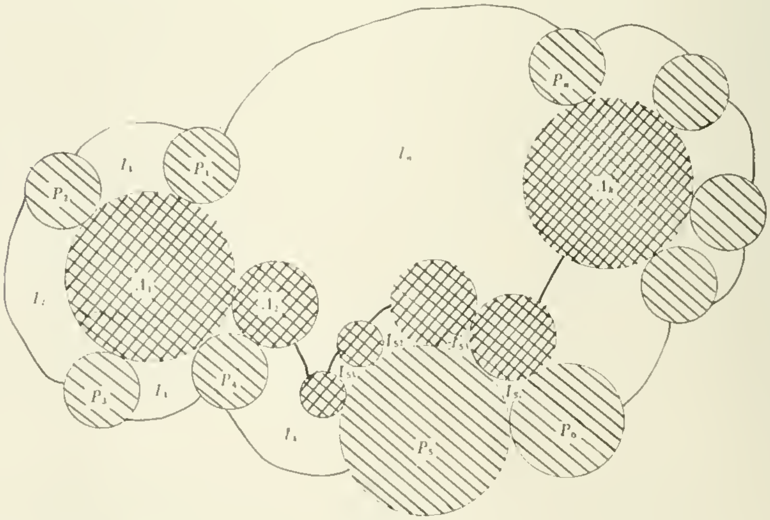

FIG. 1

Proof. First I shall prove the assertion made in the statement of the theorem to the effect that if $j<j'$, and if P_j is contiguous to A_i and $P_{j'}$ is contiguous to $A_{i'}$, then $i \leqq i'$. Suppose there exist two integers j and j' such that

$j<j'$, P_j and $P_{j'}$ are contiguous to A_i and $A_{i'}$, respectively, and $i>i'$. It follows that $i>1$ and $i'<k$. Also since P_1 is contiguous to A_1 but to no other point of γ, we see that $1<j$. Again, since P_n is contiguous to A_k but to no other point of γ, we see that $j'<n$. Thus $1<j<j'<n$; hence P_1 and $P_{j'}$ separate P_j and P_n on J. Let $A_{i_1'}$ be the point of γ with lowest subscript which is contiguous to $P_{j'}$, and let A_{i_1} be the point of γ with the highest subscript which is contiguous to P_j. Let $A_1A_{i_1'}$ denote the arc $A_1A_{i_1'}$ of A_1A_k or the point A_1 according as i_1' is not or is 1, and let $A_{i_1}A_k$ denote the arc $A_{i_1}A_k$ of A_1A_k or the point A_k according as i_1 is not or is k. Now $P_1+A_1A_{i_1'}+P_{j'}$ and $P_j+A_{i_1}A_k+P_n$ are arcs satisfying the conditions of Theorem 4. Hence $A_1A_{i_1'}$ and $A_{i_1}A_k$ have a point in common. But $1\leqq i_1'\leqq i'<i\leqq i_1\leqq k$; therefore $A_1A_{i_1'}$ and $A_{i_1}A_k$ cannot have a point in common. Thus we reach a contradiction, and we conclude that if $j<j'$, and if P_j and $P_{j'}$ are contiguous to A_i and $A_{i'}$, respectively, then $i\leqq i'$.

This result enables us to verify the following implications made in the statement of the theorem: (1) there do not exist two distinct points of γ each of which is contiguous to both P_j and P_{j+1}; (2) if no point of γ is contiguous to both P_j and P_{j+1}, then there exists an integer i such that A_i is contiguous to P_j and A_{i+1} is contiguous to P_{j+1}; and (3) if no point of β is contiguous to both A_i and A_{i+1}, then there exists an integer j such that $j<n$ and P_j and P_{j+1} are contiguous to A_i and A_{i+1}, respectively.

Suppose $I-(A_1A_k+\sum_{j=1}^{n}I_j+\sum_{j=1}^{n}\sum_{t=1}^{k_j}I_{jt})=M$ is a non-vacuous point set. Let M_1 be a component of the domain M, and let α denote the boundary of M_1. Obviously α is a subset of $J+A_1A_k$. Furthermore, α contains at least one point of J and at least one point of A_1A_k. For let P be a point of M_1, and let $P\omega$ be an arc from P to ω lying in the exterior of $(A_1A_k+P_nP_1)$. Now $P\omega$ obviously contains a point of $\alpha\cdot J$. Similarly, let PA_1 denote an arc from P to A_1 which lies in I. Now PA_1 contains a point of $\alpha\cdot A_1A_k$.

Suppose J contains two non-contiguous points X and Y of α. If X and Y do not lie together on any one of the arcs P_1P_2, P_2P_3, $\cdots$, P_nP_1, there exist two points P_r and P_s of β which separate X and Y on J; hence there exists a simple closed curve J' such that: (1) J' is a subset of $E+J+A_1A_k$; (2) $J'\cdot J=M_r+M_s$ where M_r and M_s are two mutually separated continua which contain P_r and P_s, respectively, and which separate X and Y on J; therefore $J-J'\cdot J=g_X+g_Y$ where g_X and g_Y are mutually separated segments containing X and Y, respectively; (3) J' contains at least one point O of E and at least one point of A_1A_k. If X and Y lie together on the arc P_jP_{j+1} of the set P_1P_2, P_2P_3, $\cdots$, P_nP_1, there exists a simple closed curve J' such that: (1) J' is a subset of E plus the segment XY of $P_jP_{j+1}+I_j+A_1A_k+(J-P_jP_{j+1})$; (2) $J'\cdot J$ is the sum of two mutually separated continua which separate X

and Y on J; hence $J-J'\cdot J=g_X+g_Y$ where g_X and g_Y are mutually separated segments containing X and Y, respectively; (3) J' contains at least one point O of E and at least one point of A_1A_k. Now, in either case it may be readily shown that J' separates X from Y and hence that J' separates two points X' and Y' of M_1. But this is impossible since M_1 is connected and contains no point of J'. From this contradiction we conclude that $\alpha\cdot J$ consists either of a single point or of two contiguous points. By an analogous argument it may be shown that $\alpha\cdot A_1A_k$ consists either of a single point or of two contiguous points.

Now I wish to show that α is a subset of a simple closed curve or triune J^*, which is a subset of $J+A_1A_k$, satisfying the condition that either there exists an integer j such that I_j is the interior of J^*, or there exists a pair of integers (j, t) such that I_{jt} is the interior of J^*. If $\alpha\cdot J$ is a subset of P_nP_1, then $J^*=P_nP_1+A_1A_k$. If $\alpha\cdot J$ is not a subset of P_nP_1, we shall use the following procedure.

Case 1. Suppose that P_jP_{j+1}, $(j\neq n)$, is the only arc of the set P_1P_2, $P_2P_3, \cdots, P_nP_1$ which contains $\alpha\cdot J$. It follows that if X denotes either P_j or P_{j+1}, then $P_jP_{j+1}-X$ contains at least one point Z of α. If P_j and P_{j+1} are both contiguous to the point A_i of γ, then we may show that $A_i=\alpha\cdot A_1A_k$. Suppose the contrary. Let Y denote a point of $\alpha\cdot A_1A_k$ which is different from A_i. Now Y either precedes or follows A_i, in the order A_1 to A_k. Suppose Y precedes A_i. Then $i\neq 1$ and $j\neq 1$, and Y is not contiguous to P_{j+1}. There exists a simple closed curve J' having the following properties: (1) J' is a subset of $J+A_1A_k+I_n$; (2) $J'\cdot J$ is a connected point set such that the connected point set $J-J'\cdot J$ contains $P_n+(P_jP_{j+1}-P_j)$; (3) $J'\cdot A_1A_k$ is a connected set such that $A_1A_k-J'\cdot A_1A_k$ is either a connected set containing Y and A_1, or the sum of two mutually separated connected sets, one containing Y and A_1 and the other containing A_k; (4) J' contains at least one point of I_n and at least one point of $J-P_nP_1$. By means of these four conditions imposed on J' we may readily show that J' separates Y from Z and hence that J' separates two points Y' and Z' of M_1. But this is impossible since M_1 is connected and contains no point of J'. Thus we conclude that Y cannot precede A_i on A_1A_k. Similarly Y cannot follow A_i on A_1A_k. Therefore $A_i=\alpha\cdot A_1A_k$. Thus $(A_i+P_jP_{j+1})$ is a simple closed curve or triune J^* which contains α and whose interior is I_j.

If there exists no point of γ which is contiguous to both P_j and P_{j+1}, then there exists an integer i such that P_j is contiguous to A_i and P_{j+1} is contiguous to A_{i+1}. In this case we show that $\alpha\cdot A_1A_k$ is a subset of A_iA_{i+1}. Suppose the contrary. Let Y denote a point of $\alpha\cdot A_1A_k$ not on A_iA_{i+1}. Either Y precedes A_i or Y follows A_{i+1}, in the order A_1 to A_k. Suppose Y precedes A_i.

Then $i\neq 1$ and $j\neq 1$ or n. Now there exists a simple closed curve J' having the following properties: (1) J' is a subset of $J+A_1A_k+I_n$; (2) $J'\cdot J$ is a connected point set such that the connected set $J-J'\cdot J$ contains $P_n+(P_jP_{j+1}-P_j)$; (3) $J'\cdot A_1A_k$ is connected, and $A_1A_k-J'\cdot A_1A_k$ is the sum of two mutually separated connected point sets, one containing Y and A_1, and the other containing A_{i+1} and A_k; (4) J' contains at least one point of I_n and at least one point of $J-P_nP_1$. Thus again J' separates Y from Z and hence separates two points Y' and Z' of M_1. Again we reach a contradiction, and we conclude that Y cannot precede A_i. The same argument applies to show that Y cannot follow A_{i+1}. Hence A_1A_k is a subset of A_iA_{i+1}, and $(P_jP_{j+1}+A_iA_{i+1})$ is therefore a simple closed curve J^* containing α. The interior of J^* is I_j.

Case 2. If there exists no integer r such that P_rP_{r+1} is the only arc of the set P_1P_2, P_2P_3, $\cdots$, P_nP_1 which contains $\alpha\cdot J$, it follows that $\alpha\cdot J$ is a point P_j of β. The case in which $\alpha\cdot J=P_n$ has been disposed of in the paragraph preceding Case 1. Hence suppose $j<n$. Let A_i be the point of γ with lowest subscript which is contiguous to P_j, and let A_i' be the point of γ with highest subscript which is contiguous to P_j. By means of an argument like that used in Case 1, it may be shown that no point of $\alpha\cdot A_1A_k$ precedes A_i or follows A_i' in the order from A_1 to A_k. If $i=i'$, then $\alpha\cdot A_1A_k=A_i$. Thus $(P_jP_{j+1}+A_i)$ or $(P_jP_{j+1}+A_iA_{i+1})$, according as P_j and P_{j+1} are or are not both contiguous to A_i, is a triune or simple closed curve J^* which contains α and whose interior is I_j. If $i\neq i'$, then the arc A_iA_i' of A_1A_k contains $\alpha\cdot A_1A_k$. Furthermore, since $\alpha\cdot A_1A_k$ consists either of a single point or of two contiguous points, it follows that $\alpha\cdot A_1A_k$ is a subset of one of the arcs of the set A_iA_{i+1}, $A_{i+1}A_{i+2}$, $\cdots$, $A_{i'-1}A_{i'}$. Suppose the arc $A_{i+t-1}A_{i+t}$ contains $\alpha\cdot A_1A_k$. Then since A_s, $(i\leqq s\leqq i')$, is contiguous to P_j, it follows that $(P_j+A_{i+t-1}A_{i+t})$ is a simple closed curve or triune J^* which contains α and whose interior is I_{jt}. Thus for any case it has been shown that α is a subset of a simple closed curve or triune J^* satisfying the condition that either there exists an integer j such that I_j is the interior of J^*, or there exists a pair of integers (j, t) such that I_{jt} is the interior of J^*.

Let E^* denote the exterior of J^*. Now E^* contains M_1; hence $E^*=M_1+(E^*-M_1)$, where both M_1 and E^*-M_1 are non-vacuous point sets neither of which contains a point of α. Thus E^* is the sum of two mutually separated sets, contrary to Axiom 3; and the theorem is established.

THEOREM 8. *Let the following changes, and none other, be made in the hypotheses of Theorem* 7: A_1, A_2, $\cdots$, A_k, $(k\geqq 1)$, *are points of an arc* A_1T (*in the order indicated if* $k>1$) *where* A_1T-T *is a subset of* I *and* T *is a point of* *the segment* P_nP_1, $(n\geqq 2)$. *The set* $\gamma=A_1+A_2+\cdots+A_k$ *is the set of all points of* A_1T-T *which are contiguous to points of* J; $\beta=P_1+P_2+\cdots+P_n$ *is the set of all points of* J *which are contiguous to points of* γ. *The point* P_n *is no longer restricted to be contiguous to* A_k *alone of the set* γ; I_n *denotes the interior of* P_nT+A_kT; *and* I_0 *denotes the interior of* A_1T+TP_1. *Then*

$$I = (A_1T - T) + \sum_{j=0}^{n} I_j + \sum_{j=1}^{n}\sum_{t=1}^{k_j} I_{jt}.$$

If $\beta=P_1+P_2+P_3+\cdots$ *and* $\gamma=A_1+A_2+A_3+\cdots$, *where* β *and* γ *are infinite sets each having* T *as its only limit point, then*

$$I = (A_1T - T) + \sum_{j=0}^{\infty} I_j + \sum_{j=1}^{\infty}\sum_{t=1}^{k_j} I_{jt}.$$

THEOREM 9. *Let the following changes, and none other, be made in the hypotheses of Theorem* 7: V, A_1, A_2, $\cdots$, A_k, T, $(k\geqq 1)$, *are points of the arc* VA_1T *in the order indicated, where the segment* VA_1T *is a subset of* I, *and where* V *and* T *are points of* J *in the order* V, P_1, P_2, $\cdots$, P_n, T, $(n\geqq 1)$. *The set* $\gamma=A_1+A_2+\cdots+A_k$ *is the set of all points of the segment* VA_1T *which are contiguous to points of* J, *and* $\beta=P_1+P_2+\cdots+P_n$ *is the set of all points of* J *which are contiguous to points of* γ. *The points* P_1 *and* P_n *are no longer restricted as to the number of points of* γ *to which they are contiguous. If* I_0, I_n, *and* I^* *denote the interiors of* (VP_1+VA_1), (P_nT+A_kT), *and* $(TV+VA_1T)$, *respectively, where* TV *is that arc of* J *not containing* P_1, *then*

$$I = \text{segment } VA_1T + I^* + \sum_{j=0}^{n} I_j + \sum_{j=1}^{n}\sum_{t=1}^{k_j} I_{jt}.$$

If $\beta=P_1+P_2+P_3+\cdots$ *and* $\gamma=A_1+A_2+A_3+\cdots$ *are infinite sets each having* T *as its only limit point, then*

$$I = \text{segment } VA_1T + I^* + \sum_{j=0}^{\infty} I_j + \sum_{j=1}^{\infty}\sum_{t=1}^{k_j} I_{jt}.$$

A similar formula holds for the case in which each of the points V and T is approached sequentially by a sequence from β and by a sequence from γ.

THEOREM 10. *Let the following changes be made in the hypotheses of Theorem* 7: P_1 *is contiguous to both* A_1 *and* A_k *but to no other point of* γ. *Either each of the points* A_1 *and* A_k *is contiguous to some point of* $\beta-P_1$ *or else* $k>2$. *Then either it is true that if* P_j *and* $P_{j'}$ *are contiguous to* A_i *and* $A_{i'}$, *respectively, and if* $j<j'$, *then* $i\leqq i'$, *or it is true that if* P_j *and* $P_{j'}$ *are contiguous to* A_i *and* $A_{i'}$, *respectively, and if* $j<j'$, *then* $i\geqq i'$. *If the former condition holds, if* I^* *denotes*

the interior of $A_1A_k+P_1$, *and if* I_n *denotes the point set obtained by substituting* n *for* j *in the definition of* I_j *given in the statement of Theorem* 7, *then*

$$I = A_1A_k + I^* + \sum_{j=1}^{n} I_j + \sum_{j=2}^{n} \sum_{t=1}^{k_j} I_{jt}.$$

The theorem is illustrated by Fig. 2.

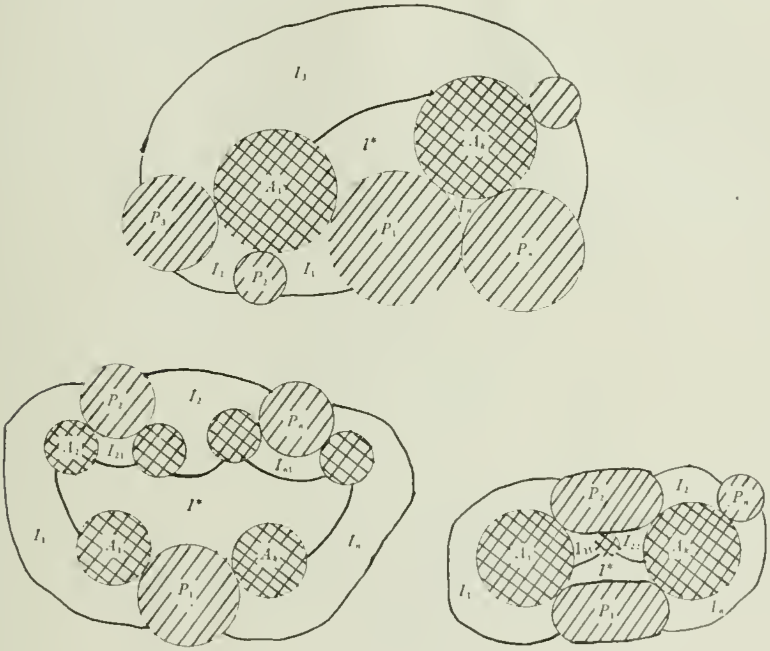

FIG. 2

If in Theorem 10, $\beta = P_1+P_2$, where P_1 and P_2 are non-contiguous, then J is the sum of two arcs, P_1XP_2 and P_1YP_2, and is such that the corresponding segments are mutually separated. Suppose $k \geqq 3$. (1) If at least one of the points A_1 and A_k, say A_1, is contiguous to P_2, then by Theorem 3, $I-A_1=D_1+D_2$, where D_1 and D_2 are the interiors of the simple closed curves $P_1XP_2+A_1$ and $P_1YP_2+A_1$, respectively. Let I_1 denote that domain above which does not contain A_k. Suppose $I_1=D_1$. Let I_2 denote the interior of $P_1YP_2+A_k$ or of $P_1YP_2+A_{k-1}A_k$, according as A_k is or is not contiguous to P_2. (2) If neither A_1 nor A_k is contiguous to P_2, then $I-A_1A_2=D^*+D^{**}$, where D^* and D^{**} are the interiors of $P_1XP_2+A_1A_2$ and $P_1YP_2+A_1A_2$, respectively. Let I_1 denote the domain above which does not contain A_k. Suppose $I_1=D^*$. Let I_2 denote the interior of $P_1YP_2+A_{k-1}A_k$. Then in either of these two cases we have, in accordance with the above notation, the following theorem:

THEOREM 11. $I = A_1A_k + I^* + I_1 + I_2 + \sum_{t=1}^{k_2} I_{2t}$.

This theorem is illustrated by Fig. 3.

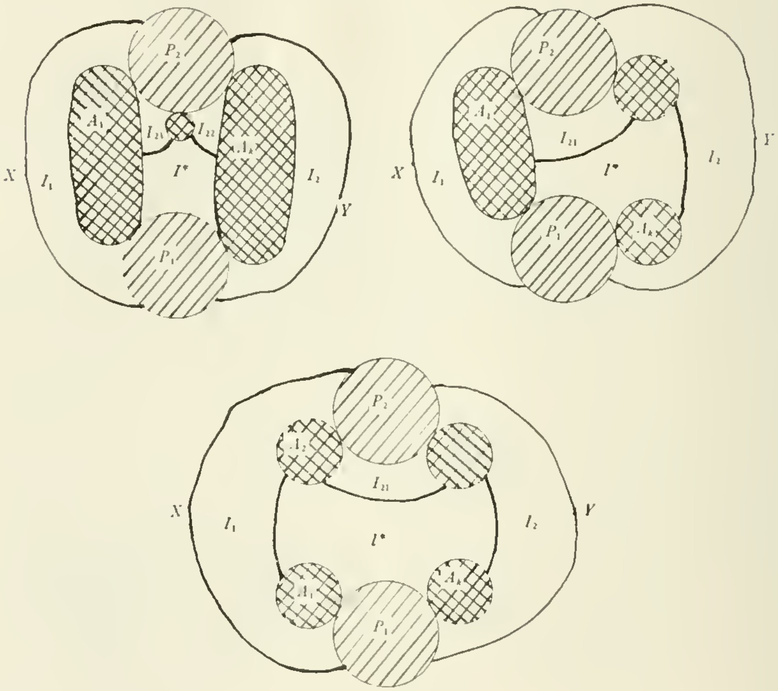

FIG. 3

THEOREM 12. *Let J and C denote two point sets, each of which is either a simple closed curve or a triune. Let I and D denote the interiors of J and C, respectively, and suppose P is a point of $I \cdot D$. Then there exists a simple closed curve or triune Q such that* (1) *Q is a subset of $J+C$, and* (2) *the interior of Q contains P and is a subset of $I \cdot D$.*

Indication of proof. If at least one of the two sets J and C is a triune, one of the sets is a subset of the other plus its interior and hence will have the properties required of Q.

A similar situation exists if both J and C are simple closed curves and one of them is a subset of the other plus its interior.

If neither of the simple closed curves J and C is a subset of the other plus its interior, let A be a point of $J \cdot D$, and let PA be an arc from P to A lying wholly in D. Let O denote the first point, in the order P to A, that PA has in common with J. I wish to show the existence of a simple closed curve or triune Q' such that (1) Q' is a subset of $C+J\cdot D$, (2) $Q'\cdot C$ is connected, (3) $Q'\cdot J\cdot D$ is connected and contains O, and (4) the interior of Q' contains $(PO-O)$. Obviously, the existence of Q' will be established if it can be shown that there exists a point set K (which shall contain $Q'\cdot J\cdot D$) which is a connected subset of J containing O and which satisfies with respect to C those conditions satisfied by any one of the point sets P, A_1A_k, segment AXB, (A_1T-T), segment VA_1T, A_1A_k, or A_1A_k with respect to the corresponding given simple closed curve of Theorems 6, 7, 3, 8, 9, 10, and 11, respectively. There are several cases.

Case 1. Suppose O is not contiguous to any point of C. Let B denote a point of $J\cdot E$. The set J is the sum of two arcs from O to B, say OXB and OYB. Let M denote the closed point set consisting of C together with all points Z of D such that Z is contiguous to at least one point of C. Let A_1 and A_2 denote the first points that OXB and OYB have in common with M, in the order O to B. If both A_1 and A_2 are points of C, the hypothesis of Theorem 3 is satisfied, and the segment A_1OA_2 is the desired point set K.

Case 2. Suppose O is not contiguous to any point of C and that one of the points A_1 and A_2 described in Case 1, say A_1, is a point of D, while the other, A_2, is a point of C. If A_1 is contiguous to two or more points of C, then the hypotheses of Theorem 8 are satisfied, and $(A_1OA_2-A_2)$ is the desired point set K. If A_1 is contiguous to only one point F of C, and F is not contiguous to A_2, then the hypotheses of Theorem 3 are satisfied, and $(A_1OA_2-A_2)$ is the desired point set K. If F is contiguous to A_2, let A_1' denote the first point of OXB, in the order O to B, which is either a point of D that is contiguous to a point of $C-F$ or else a point of C. Now if A_1' is a point of D, the hypothesis of Theorem 8 is satisfied, and $(A_1'OA_2-A_2)$ is the desired point set K. If A_1' is a point of C, the hypothesis of Theorem 9 is satisfied, and the segment A_1OA_2 is the desired point set K.

The remaining cases may be treated with similar methods.

If Q' is a triune, Q' has the properties required of Q, since the interior of Q' cannot contain any points of J. (No simple closed curve may contain a point of each complementary domain of a triune.)

If Q' is a simple closed curve whose interior contains no point of J, then $Q'=Q$. If the interior of Q' contains any points of J, let M denote the set of all such points of J. Now, $M+Q'$ contains a simple closed curve Q having the required properties. The proof of this last statement is little different from the proof for spaces in which there do not exist contiguous points.

As an immediate result of Theorem 12 we have the following theorem:

THEOREM 13. *Let* $J_1, J_2, \cdots, J_n$ *denote* n *point sets each of which is either a simple closed curve or a triune; and suppose that* $I_1, I_2, \cdots, I_n$, *the interiors of* $J_1, J_2, \cdots, J_n$, *respectively, have a point* O *in common. Then there exists a simple closed curve or triune* J *which is a subset of* $J_1+J_2+\cdots+J_n$ *and is such that* I, *the interior of* J, *is a subset of* $I_1\cdot I_2\cdot\cdots\cdot I_n$ *and contains* O.

THEOREM 14. *Suppose each of the sets* J_1 *and* J_2 *is either a simple closed curve or a triune. Let* I_1 *and* I_2 *denote the interiors of* J_1 *and* J_2, *respectively, and suppose* I_1+I_2 *is connected. Then there exists a simple closed curve or triune* J *which is a subset of* J_1+J_2 *and whose interior is a subset of* $I_1\cdot I_2$.

Indication of proof. Assume the theorem false. Then by Theorem 12 $I_1\cdot I_2=0$. Hence there exist points P_1 and P_2 of I_1 and I_2, respectively, such that P_1 is contiguous to P_2. Thus P_1 is a point of J_2 and P_2 is a point of J_1. There exists a point F of $J_1\cdot J_2$ and an arc P_1F of J_2 such that P_1F-F is a subset of I_1. Let P_2F denote an arc of J_1. The point set P_1F+P_2F contains a simple closed curve or triune J' containing P_1 and P_2, whose interior I' is a subset of I_1. The set $J'+I'\cdot J_2$ contains a simple closed curve or triune J satisfying the required conditions. Thus the assumption is false, and the theorem is established.

With the help of Theorems 12 and 14, the following theorem may be established:

THEOREM 15. *Let* J_1 *and* J_2 *denote two point sets, each of which is either a simple closed curve or a triune. Let* I_1 *and* I_2 *denote the interiors of* J_1 *and* J_2, *respectively. Suppose* I_1+I_2 *is connected. Then there exists a simple closed curve or triune* J *which is a subset of* J_1+J_2, *and whose interior contains* I_1+I_2.

THEOREM 16. *Let* $J_1, J_2, \cdots, J_n$ *denote* n *point sets each of which is either a simple closed curve or a triune. Let* $I_1, I_2, \cdots, I_n$ *denote the interiors of* $J_1, J_2, \cdots, J_n$, *respectively. Suppose* $I_1+I_2+\cdots+I_n$ *is connected. Then there exists a point set* J *which is either a simple closed curve or a triune, which is a subset of* $J_1+J_2+\cdots+J_n$, *and which is such that* I, *the interior of* J, *contains* $I_1+I_2+\cdots+I_n$.

THEOREM 17. *If* J *is a simple closed curve or triune, then* I, *the interior of* J, *contains at least one point which is not contiguous to any point of* J.

Proof. Suppose each point of I is contiguous to some point of J. Let P be any point of I. I wish to show the existence of two distinct points of I, each contiguous to P. Since I is connected and contains infinitely many points, P is either contiguous to a point of I or is a limit point of I. By Axiom C, J contains all limit points of I; hence P must be contiguous to a point Q of I. In case there exists a point P_1 of J which is contiguous to both

P and Q, then I_1, the interior of the triune PQP_1, is a connected subset of I. Since J contains all the limit points of I_1, it follows that P is contiguous to a point T of I_1. Thus for this case there exist two distinct points (Q and T) of I, each contiguous to P. In case no point of J is contiguous to both P and Q, let P_1 and Q_1 denote points of J which are contiguous to P and Q, respectively, and let P_1Q_1 denote an arc of J. The point set P_1Q_1+P+Q contains a simple closed curve J_2, which contains P and Q. Let I_2 denote the interior of J_2. The argument used above may be used here to show that P is contiguous to a point T of I_2.

Let QT denote an arc lying in I. The point set $QT+P$ contains a simple closed curve or triune J' whose interior I' contains infinitely many points, no one contiguous to any point of J. Thus we reach a contradiction, and the theorem is established.

Example 1. In the euclidean plane let C_1, C_2, and C_3 denote three circles, each tangent externally to each of the others. Denote their centers by P_1, P_2, and P_3, respectively, and their radical center by O. Let $A_1=C_1\cdot C_2$, $A_2=C_2\cdot C_3$, and $A_3=C_3\cdot C_1$. Let B_1, B_2, and B_3 denote points on the rays OA_1, OA_2, and OA_3, respectively, such that $d(O, B_1)=d(O, B_2)=d(O, B_3)$, and $d(O, B_1)>d(O, A_1)$. Let B_3B_1, B_1B_2, and B_2B_3 be circular arcs having P_1, P_2, and P_3, respectively, as centers and lying on the non-O sides of the lines B_3B_1, B_1B_2, and B_2B_3, respectively. Let $\beta_{11}=B_3B_1+OB_3+OB_1$, $\beta_{21}=B_1B_2+OB_1+OB_2$, $\beta_{31}=B_2B_3+OB_2+OB_3$. For each point P of β_{i1} and each j, $(i=1, 2, 3; j=1, 2, 3, \cdots)$, let Q_{Pij} denote the point of the interval P_iP such that $d(P_i, Q_{Pij})=d(P_i, A_i)+[d(P_i, P)-d(P_i, A_i)]/j$, and let β_{ij} denote the set of all points Q_{Pij}. Let T_i, $(i=1, 2, 3)$, denote C_i plus its interior. Let S denote the following collection: (1) T_i, $(i=1, 2, 3)$, is an element of S, and (2) each point of the plane not in $T_1+T_2+T_3$ is an element of S. For each positive integer n, let G_n denote the following subsets of S: (1) the interior of each circle of radius $1/n$ or less which neither encloses nor contains a point of $T_1+T_2+T_3$ is an element of G_n; (2) for each pair (i, j), $(i=1, 2, 3; j=n, n+1, \cdots)$, the set consisting of T_i and all elements of S enclosed by β_{ij} is an element of G_n. If each element of S is called a "point," each element of G_1 is called a "region," and each of the "points" T_1, T_2, and T_3 is "contiguous to" each of the others, then each of the axioms of this paper is nonvacuously satisfied.

Example 2. In a euclidean space let s_1, s_2, s_3, s_4, and s_5 denote the spheres whose equations are $(x-1)^2+y^2+z^2=3/2^2$, $(x+1/2)^2+(y-(3)^{1/2}/2)^2+z^2=3/2^2$, and $(x+1/2)^2+(y+(3)^{1/2}/2)^2+z^2=3/2^2$, $x^2+y^2+(z-(3)^{1/2}/12)^2=3/12^2$, and $x^2+y^2+(z+(3)^{1/2}/12)^2=3/12^2$, respectively. Each of these spheres is tangent externally to each of the others. For each i, $(i=1, \cdots, 5)$, let T_i denote s_i

plus its interior. Let S denote the collection defined as follows: (1) For each i, $(i=1, \cdots, 5)$, T_i is an element of S. (2) Let K denote the (x, y)-plane. Each point of the unbounded component of $K-K(T_1+T_2+T_3)$ is an element of S. (3) For each set of three distinct positive integers i, j, and k, such that each integer is less than 6 and at least one integer is either 4 or 5, let M_{ijk} denote the plane which contains the centers of the spheres s_i, s_j, and s_k. Each point of the bounded component of $M_{ijk}-M_{ijk}(T_i+T_j+T_k)$ is an element of S. For each positive integer n, let G_n denote the subsets of S defined as follows: (1) The interior of each circle of radius $1/n$ or less which is a subset of $S \cdot M_{ijk}$ or of $S \cdot K$ is an element of G_n. (2) In each of the sets $S \cdot M_{ijk}$, construct three sequences of segments like the three sequences of segments constructed within the triune of Example 1, and in $S \cdot K$ construct three sequences of segments like those constructed in the exterior of the triune of Example 1. For each pair (i, j), $(i=1, \cdots, 5; j=n, n+1, \cdots)$, the subset of S consisting of T_i together with all elements of $S-(T_1, T_2, \cdots, T_5)$ each of which is enclosed by T_i plus the jth segment of one of the sequences constructed is an element of G_n. If each element of S is called a "point," each element of G_1 is called a "region," and each of the "points" $T_1, \cdots, T_5$ is "contiguous to" each of the others, then all the axioms of this paper are non-vacuously satisfied.

Example 2 shows that Theorem 6 fails to hold if J denotes a triune instead of a simple closed curve.

THEOREM 18. *If J is a simple closed curve, if I is the interior of J, and if H and K are mutually exclusive compact continua lying in $J+I$, then no two points of H separate two points of K on J.*

Proof. Suppose on the contrary that the points A and B of H separate the points C and D of K on J. It follows from Theorem 24 of S.C.P. that K contains an irreducible continuum T from $K \cdot ACB$ to $K \cdot ADB$. By Theorem 28 of S.C.P., $T-T \cdot J$ is a connected set having boundary points C_1 and D_1 in segments ACB and ADB, respectively. It may be readily shown that the component of $S-(J+H)$ which contains $T-T \cdot J$ contains a segment C_2YD_2, where C_2 and D_2 are points of segments ACB and ADB, respectively, and neither point belongs to H. Thus the arc C_2YD_2 contains no point of H and lies, except for its end points, in I.

The above argument may be repeated to show that there exists an arc A_2XB_2 which contains no point of C_2YD_2, where the segment A_2XB_2 lies in I, and A_2 and B_2 separate C_2 and D_2 on J. But this contradicts Theorem 4.

THEOREM 19. *The interior of a simple closed curve or triune is not a subset of any simple closed curve or triune.*

Proof. Let I denote the interior of a simple closed curve or triune J. By Theorem 5, I is not a subset of a triune. Suppose I is a subset of a simple closed curve C. By Theorem 17, there exists a point P of I which is not contiguous to any point of J. Now P is either a limit point of $S-C$ or is contiguous to a point of $S-C$. If P is a limit point of $S-C$, let R be a connected domain containing P, but containing no point of J, and let Q be a point of $R\cdot(S-C)$. If P is not a limit point of $S-C$, let Q be a point of $S-C$ which is contiguous to P. In either case, Q belongs to $I\cdot(S-C)$.

THEOREM 20. *If neither of the contiguous points X and Y separates the point A from the point B, then their sum does not separate A from B.*

Proof. Suppose $S-(X+Y)=S_A+S_B$, where S_A and S_B are mutually separated sets containing A and B, respectively. There exists an arc α from A to B which does not contain Y. The arc α contains X; hence α contains an arc AX. Similarly, there exists an arc AY. The point set $AX+AY-(X+Y)$ is connected and hence lies in S_A. The set $AX+AY$ contains a simple closed curve or triune J which contains X and Y and is a subset of S_A+X+Y. Let Q be any point of $J-(X+Y)$, and let I and E denote the two complementary domains of J. Since Q is a point of S_A and is a boundary point of each of the connected sets I and E, it follows that I and E are subsets of S_A. Hence S is a subset of S_A+X+Y. Thus we reach a contradiction, and the theorem is established.

AXIOM 4. *If P and Q are two distinct non-contiguous points, there exists a simple curve or triune which separates P from Q.*

With the help of Axiom 4, the Borel-Lebesgue theorem (Theorem 5 of S.C.P.), and Theorem 16, the next theorem may be established.

THEOREM 21. *If H and K are two mutually separated compact continua, there exists a simple closed curve or triune which separates H from K.*

THEOREM 22. *If J is a simple closed curve or triune containing the contiguous points A and B, I is the interior of J, and P is a point of $J+I$ which is not contiguous to A, then there exists a simple closed curve or triune J^* satisfying the following conditions*: (1) *It is a subset of $J+I$.* (2) *Its intersection with J is an arc containing A and B.* (3) *Its exterior contains P.*

Proof. Suppose the theorem is false. If may be readily shown that there exists a simple closed curve or triune which satisfies conditions (1) and (2). Furthermore it may be readily shown that if P is any point of J not contiguous to A, or if P is a point of I not contiguous to A such that there is a simple closed curve or triune J' containing P and satisfying conditions (1) and (2), then there exists a simple closed curve or triune satisfying conditions (1), (2), and (3). Hence our supposition implies that P is a point of I, and every simple closed curve or triune, satisfying conditions (1) and (2), encloses P. Let PX denote an arc from P to some point X of $J-(A+B)$ such that $PX-X$ is a subset of I. Let S_1 denote the set consisting of P and all points Y of $PX-X$ such that every simple closed curve or triune satisfying conditions (1) and (2) encloses the interval PY of PX, and let S_2 denote $PX-S_1$. By the Dedekind-cut proposition (P.S.T., chap. 1, Theorem 64) there exists a point Q which is either the last point of S_1 or the first point of S_2, in the order P to X.

Suppose Q is the first point of S_2. There exists a simple closed curve or triune J' satisfying the following conditions: If $Q=X$, then $J'=J$; if $Q\neq X$, then J' contains Q, satisfies conditions (1) and (2), and encloses $PQ-Q(=S_1)$. There exists a third simple closed curve or triune J_1, which is a subset of J' plus its interior and is such that $J'\cdot J_1$ is an arc containing A and B but not containing Q. Thus Q is exterior to J_1; hence $PQ-Q$ either intersects J_1 or is exterior to J_1. Both these possibilities are ruled out since J_1 satisfies conditions (1) and (2) and hence encloses $PQ-Q$. Thus Q is not the first point of S_2.

Suppose Q is the last point of S_1. By means of an argument analogous to that used in the last paragraph, it may be shown that Q is not contiguous to a point of S_2. Thus Q is a limit point of S_2. Now Q cannot be contiguous to both A and B; for if such were the case, the triune ABQ would satisfy conditions (1) and (2) and hence would enclose $PQ=S_1$ and, in particular, the point Q. Let C denote a point of the pair (A, B) which is not contiguous to Q, and let $C\omega$ be an arc from C to ω lying in J plus its exterior and containing no point of J which is contiguous to Q. By Theorem 21, there exists a simple closed curve or triune J' which separates Q from $C\omega$. Since Q is a limit point of S_2, it follows that I', the interior of J', contains a segment QW of S_2. There exist a point F of the segment QW, and a simple closed curve or triune J_1 which satisfies conditions (1) and (2) and is such that F is in E_1, the exterior of J_1. Let I_1 denote the interior of J_1. By Theorem 12, there exists a simple closed curve or triune J_2, which is a subset of J_1+J' and whose interior I_2 contains Q and is a subset of $I_1\cdot I'$. There exists a point T of $J_2\cdot I'\cdot J_1$. It may be readily shown that the point set $J_1-T+(J_2-T)$ contains a simple closed curve or triune J^* which satisfies conditions (1) and (2). Furthermore, since I_2+T+E_1 is a connected point set containing Q and ω but no point of J^*, Q is exterior to J^*. Hence Q is not a point of S_1.

Thus we have reached a contradiction and the theorem is established.

THEOREM 23. *The interior of every triune is non-compact and so is the interior of every simple closed curve which contains two contiguous points.*

Proof. Let J denote a simple closed curve or triune containing the contiguous points A and B, and let I denote the interior of J. Suppose that I is compact. It follows that $J+I$ is compact, completely separable, and metric. Let D_1, D_2, D_3, $\cdots$ be a sequence of domains which properly covers $J+I$ and with respect to which $J+I$ is completely separable. Let n_1 denote the smallest integer (which exists in view of Theorems 17 and 22) such that there exists a simple closed curve or triune J_1 which is a subset of $J+I$, and such that $J\cdot J_1$ is an arc containing A and B, and D_{n_1} is exterior to J_1. Let n_1, n_2, n_3, $\cdots$ be an increasing sequence of positive integers satisfying the following conditions: For each integer $k>1$, n_k is the smallest integer greater than n_{k-1} such that there exists a simple closed curve or triune J_k which is a subset of J_{k-1} plus its interior, and such that $J_{k-1}\cdot J_k$ is an arc containing A and B, and D_{nk} is exterior to J_k. For each integer r, let α_r denote the point $J_r-(A+B)$ in case J_r is a triune. Otherwise, let α_r denote an arc P_rQ_r of $J_r-(A+B)$, where Q_r is either contiguous to B or else $d(Q_r, B)<1/r$, and where P_r is either contiguous to A or else $d(P_r, A)<1/r$. For each integer n let E_n denote the exterior of J_n. For each integer r and each point P of α_r there exists an integer n_P such that E_{n_P} contains P. By the Borel-Lebesgue theorem, there exists a finite collection of these domains covering α_r. The domain of this finite collection with greatest subscript E_s, contains all other domains of the finite collection; therefore E_s contains α_r. Hence $\alpha_r\cdot J_s=0$, and consequently $\alpha_r\cdot\alpha_s=0$. Thus there exists an increasing sequence of integers r_1, r_2, r_3, $\cdots$ such that for each integer $k>0$, $\alpha_{r_k}\cdot\alpha_{r_{k+1}}=0$; hence α_{r_1}, α_{r_2}, $\cdots$ is a sequence of mutually exclusive continua. There exists a subsequence of this sequence which converges to a sequential limiting set L containing A and B. Hence by Theorem 33 of S.C.P., L is a perfect continuum and is therefore uncountable. For each integer n, L is a subset of J_n plus its interior. Hence if L contains a point X other than A or B, then X is contiguous to A or B. But the set of all such points X is at most countable by Theorem 14 of S.C.P. Hence L is countably infinite or finite. We have thus reached a contradiction, and the theorem is established.

THEOREM 24.* *If H and K are two mutually separated, closed and compact point sets containing the points A and B, respectively, then there exists a simple closed curve or triune J which separates A from B such that $J\cdot(H+K)=0$.*

Proof. Let h_A and k_B denote the components of H and K containing A and B, respectively. From Theorem 21 it follows that for each component h of H there exists a simple closed curve or triune J_{hB} which separates h from

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* Cf. L. Zoretti, *Sur les fonctions analytiques uniformes*, Journal de Mathématiques Pures et Appliquées, vol 1 (1905), pp. 9–11.

k_B. Let D_{hB} denote that complementary domain of J_{hB} which contains h. The collection of all such domains D_{hB} covers H; hence there exists a finite collection T of such domains covering H. Let T^* denote the point set which is the sum of all the elements of T. By Theorem 16 there exists a simple closed curve or triune J_{AB} whose interior with respect to B contains that component of T^* which contains A. Furthermore, $(H+k_B)\cdot J_{AB}=0$. For each component k of K let J_{Ak} denote a simple closed curve or triune (which exists in view of the preceding argument) which separates A from k and is such that $(H+K)\cdot J_{Ak}=0$. Let D_{Ak} denote the interior of J_{Ak} with respect to A. The collection of all such domains D_{Ak} covers K; hence there exists a finite collection G of such domains covering K. Let G^* denote the sum of the elements of G. By Theorem 16 there exists a simple closed curve or triune J whose interior with respect to A contains that component of G^* which contains B. Furthermore, $(H+K)\cdot J=0$.

THEOREM 25.[†] *If H and K are two mutually separated closed and compact point sets, and if neither H nor K separates the point A from the point B, then $H+K$ does not separate A from B.*

Proof. Suppose $H+K$ separates A from B.

Case 1. Suppose H is connected. With the help of Theorem 24, it may be shown that there exists a domain D which contains K but no point of H and is the sum of a finite number of components each of which is bounded by a simple closed curve or triune containing no point of $H+K$. Let AXB denote an arc from A to B containing no point of K, and AYB an arc containing no point of H. Let $J_1, J_2, J_3, \cdots, J_r$ denote the boundaries of those components of D which have points in common with AYB. There are four possibilities: (1) A and B are both in $S-D$, (2) A and B are the same component of D, (3) A and B are in different components of D, (4) A or B is in $S-D$ while the other is in D. In each case it may be readily shown that $(AYB-AYB\cdot D)+J_1+J_2+\cdots+J_r+AXB\cdot D$ contains an arc from A to B.

Case 2. Suppose H has only a finite number of components, $h_1, h_2, \cdots, h_n$. By Case 1, h_1+K does not separate A from B. The sets h_2 and h_1+K are mutually separated; hence by Case 1, $h_2+(h_1+K)$ does not separate A from B. Continuing this process we obtain the fact that $h_n+(h_{n-1}+\cdots+h_1+K)$ does not separate A from B.

Case 3. Suppose H has infinitely many components. Let AXB denote an arc containing no point of K. By Axiom C, at most a finite number of components of K contain points which are contiguous to points of AXB. Let

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[†] Cf. P. Alexandroff, *Sur les multiplicités cantoriennes et le théorème de Phragmèn-Brouwer généralisé*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 183, pp. 722–724.

$k_1, k_2, \cdots, k_n$ be the set of all such components of K. By Case 2 there exists an arc AYB which contains no point of $H+k_1+\cdots+k_n$. If $AYB\cdot K=0$, the theorem is established. Suppose $AYB\cdot K\neq 0$. For each component k of K which intersects AYB, the set $H+AXB+K$ is the sum of two mutually separated sets, one containing A and the other containing k, by Theorem 27 of S.C.P. By Theorem 24 there exists a simple closed curve or triune J_k which separates A from k and is such that $J_k\cdot(H+AXB+K)=0$. Let I_k denote the interior of J_k with respect to A. The collection of all such domains I_k covers the closed point set $AYB\cdot K$; hence there exists a finite subcollection T doing so. Let T^* denote the sum of the elements of T. By Theorem 16, for each component t of T^* there exists a simple closed curve or triune C_t which is a subset of the boundary of t and whose interior with respect to A contains t. Let $C_1, C_2, \cdots, C_s$ be the set of all simple closed curves or triunes thus obtained. The point set $(AYB-AYB\cdot T^*)+C_1+C_2+\cdots+C_s$ contains an arc from A to B which contains no point of $H+K$.

THEOREM 26. *If J is a simple closed curve, I is a complementary domain of J, H and K are two mutually separated subcontinua of J, α and β are the two components of $J-(H+K)$, and C is a simple closed curve which separates H from K, then there exists an arc AXB such that A and B are points of α and β, respectively, and segment AXB is a subset of $I\cdot C$.*

Proof. Suppose the theorem is false. Let ω be a point of $S-(J+I+C)$, which exists by Theorem 19. Let D denote the interior of C. One of the sets H and K, say H, is a subset of D. Obviously, C contains at least one point of each of the sets α, β, and I. Let P, Q, Z, and W denote points of $\alpha\cdot C$, $\beta\cdot C$, H, and K, respectively. Let α' and β' denote continua which are subsets of α and β, respectively, and which contain $\alpha\cdot C$ and $\beta\cdot C$, respectively. Now, $\alpha'+\beta'+I\cdot C$ is the sum of two mutually separated continua M_1 and M_2 containing α' and β', respectively. Let E denote the last point of the arc PZQ, in the order P to Q, which is either a point of M_1 or contiguous to a point of M_1; and let F denote the first point of PZQ which is either a point of M_2 or contiguous to a point of M_2. It may be readily seen with the help of Theorem 4 that if E and F are distinct, then E precedes F in the order P to Q. Suppose first that E and F are distinct non-contiguous points. The interval EF of PZQ contains at least one point of H. Furthermore, there exists an arc OL where L is a point of segment EF and $OL-L$ is a subset of $I-I\cdot C$. By Theorem 12 there exists a simple closed curve J' which is a subset of $J+C$ and contains EF, and whose interior I' is a subset of $I\cdot D$ and contains $OL-L$. Now J' contains no point of K, for otherwise $H+I'+K$ would be a connected set containing no point of C. Thus J' contains an arc $E'F'$, where E' and F'

are points of segments LEW and LFW, respectively, and the segment $E'F'$ is a subset of $I\cdot C$. Thus segment $E'F'$ is a subset of M_1 and also a subset of M_2. From this contradiction, it follows that E and F are identical or contiguous. Also $(E+F)\cdot(M_1+M_2)=0$. Similarly, if T denotes the last point of PWQ which is either a point of M_1 or contiguous to a point of M_1, and if V denotes the first point of PWQ which is either a point of M_2 or contiguous to a point of M_2, then T and V are identical or contiguous. Let C' denote a simple curve which separates M_1 from M_2. Let $P'Q'$ denote an arc such that P' and Q' are points of M_1 and M_2, respectively, and segment $P'Q'$ is a subset of $J+I-(J+I)\cdot(C+E+F+T+V)$. Let G denote a point of segment $P'Q'$ which belongs to C', and let XY denote an arc which is a subset of C' and contains G, where X and Y are points of J and segment XY is a subset of I. Now X and Y do not lie in different components of $J-(\alpha'+\beta')$, for otherwise, $H+K+XY$ would be a connected set containing no point of C. Suppose both X and Y are points of that component of $J-(\alpha'+\beta')$ which contains H. If either X or Y preceded E in the order PZQ, we would have two arcs (one a subset of $C\cdot M_1+E$, the other a subset of $C\cdot M_2+$interval $Q'G$ of arc $P'Q'+$interval XG of arc XY) satisfying the hypothesis of Theorem 4 but not the conclusion of this theorem. Similarly, it may be shown that neither X nor Y follows F. Hence X is identical with Y or contiguous to Y. Again a contradiction is reached, and the theorem is established.

THEOREM 27. *If A, X, B, and Y are points of the simple closed curve J in the order indicated, if I is the interior of J, and if H and K are mutually separated closed and compact subsets of $J+I$ such that $H\cdot AXB=0$ and $K\cdot AYB=0$, then there exists an arc AB lying in $J+I-(H+K)$.*

Proof. Two cases will be considered.

Case 1. Suppose there exists no component T of $H+K$ such that $T+J-(A+B)$ is connected. Let X_1 and X_2 denote continua which are subsets of segments AYB and AXB, respectively, and which contain $H\cdot AYB$ and $K\cdot AXB$, respectively; and let α and β denote the components of $J-(X_1+X_2)$ which contain A and B, respectively. By Theorem 27 of S.C.P., X_1+X_2+H+K is the sum of two mutually separated closed sets M_1 and M_2 containing X_1 and X_2, respectively. Hence by Theorem 24 there exists a simple closed curve C which separates X_1 from X_2 and contains no point of M_1+M_2. Thus C contains no point of $H+K$. By Theorem 26 there exists an arc $A'X'B'$, where A' and B' are points of α and β, respectively; and segment $A'X'B'$ lies in $I\cdot C$. Now $A'X'B'+\alpha+\beta$ contains an arc AB lying in $J+I-(H+K)$.

Case 2. Suppose that for at least one component T of $H+K$, $T+J$

$-(A+B)$ is connected. With the help of Axiom C, it may be shown that there is only a finite number of such components. Either H or K, say H, contains at least one such component. There exists a point Y_1 which is the first point of segment AYB, in order A to B, which is either a point of, or contiguous to a point of, such a component of H.

First suppose there exists at least one point Q of AY_1-A having the property that there is a component T_Q of K such that $Q+T_Q+$segment AXB is connected. Let $Q_1, Q_2, \cdots, Q_n$ denote all such points in the order A to Y_1. Obviously Q_i, $(i=1, 2, \cdots, n)$, is not a point of $H+K$. If the segment AQ_1 of AY_1 contains no point of H, let α_1 denote the interval AQ_1. If segment AQ_1 contains a point of H, it may be shown with the help of Theorem 18 and Case 1 of this theorem that there exists an arc α_1, from A to Q_1, lying in $J+I-(H+K)$. Similarly, if $n>1$, there exists an arc α_i, $(i=2, 3, \cdots, n)$, from Q_{i-1} to Q_i, lying in $J+I-(H+K)$. The set $\alpha_1+\alpha_2+\cdots+\alpha_n$ contains an arc α from A to Q_n. Now let X_2 denote the last point of segment AXB, in order A to B, for which there exists a component T_2 of H, such that $Y_1+T_2+X_2$ is connected. If $Q_n=Y_1$, and if there exists a point P of AXB between X_2 and B for which there is a component T' of K such that $Y_1+T'+P$ is connected, then, denoting Q_n by Q', we have an arc $AQ'=\alpha$ having the following properties: (1) The arc AQ' lies in $J+I-(H+K)$, (2) Q' is a point of segment AYB and does not precede Y_1, (3) there is at least one component T of H for which $[T+\text{segment } AXB+\text{interval } AQ' \text{ (of } AYB) -A]$ is connected, but $T+Q'B-Q'$ is not connected, and (4) there exists no component W of $H+K$ such that $W+AYB-Q'$ is connected. If $Q_n=Y_1$, and if there exists no such point P, let X_1 denote the first point of AXB, such that $Y_1+T_2+X_1$ is connected; or if $Q_k\neq Y_1$, let X_1 denote the first point of AXB for which there exists a component T of H such that Y_1+T+X_1 is connected. In either case, X_1 is a point of segment AXB, and with the help of Theorem 18, Case 1, it may be shown that there exists an arc β from Q_n to X_1 lying in $J+I-(H+K)$. If $X_1\neq X_2$, then by means of an argument like that used in obtaining α, it may be shown that there exists an arc γ, from X_1 to X_2, lying in $J+I-(H+K)$. If $X_1=X_2$, let γ denote X_1. If there exists no point X' of AXB between X_2 and B for which there is a component T of K such that segment $AYB+T+X'$ is connected, then the argument for obtaining α may be used here to obtain an arc δ, from X_2 to B, lying in $J+I-(H+K)$. In this case, $\alpha+\beta+\gamma+\delta$ contains an arc from A to B, and the theorem is established. If there exists a point X', as described, let Q' denote the first point of segment AYB for which exists a component T of K, such that $Q'+T+$segment X_2B is connected. It follows from Theorem 18 and Case 1, that there exists an arc η, from X_2 to Q', lying in $J+I-(H+K)$. In this case, $\alpha+\beta+\gamma+\eta$ contains an arc AQ' having the four properties listed above.

Now suppose there is no point Q as described above. Let X_2 have the same meaning as above, and let γ denote an arc from A to X_2 lying in $J+I-(H+K)$. Again, if there exists no point X', as described above, let δ denote an arc from X_2 to B lying in $J+I-(H+K)$. In this case $\gamma+\delta$ contains an arc from A to B, and the theorem is established. If there exists a point X', as described above, let Q' and η be defined as before. Now $\gamma+\eta$ contains an arc AQ' with properties listed above.

Thus, in any case either there exists an arc AB lying in $J+I-(H+K)$, or else there exists an arc AQ' having the properties listed above. In the latter case, if there exists no point Y' of AYB, between Q' and B, for which there exists a component T of H such that $Y'+T+$segment AXB is connected, then the argument for obtaining α may be used to obtain an arc $Q'B$ lying in $J+I-(H+K)$. In such a case, $AQ'+Q'B$ contains an arc from A to B, and the theorem is established. If there exists a point Y', as described above, let Y_2 denote the first such point in the order Q' to B. Now the argument for obtaining AQ' may be repeated to obtain either an arc $Q'B$ lying in $J+I-(H+K)$ (in which case $AQ'+Q'B$ contains an arc from A to B, and the theorem is established), or an arc $Q'Q''$ of such nature that $AQ'+Q'Q''$ contains an arc AQ'' having the following properties: (1) AQ'' lies in $J+I-(H+K)$, (2) Q'' is a point of segment AYB and does not precede Y_2, (3) there are at least two components of H such that if T denotes either of them, then $[T+$segment $AXB+$interval AQ'' of $AYB-A]$ is connected, but $[T+Q''B-Q'']$ is not connected, and (4) there exists no component W of $H+K$, such that $W+AYB-Q''$ is connected. Since there is only a finite number of components of H such that if T denotes any one of them, $T+J-(A+B)$ is connected, the above process may be repeated until we have obtained either an arc AB lying in $J+I-(H+K)$, or an arc $AQ^{(k)}$ lying in $J+I-(H+K)$, where $Q^{(k)}$ is a point of segment AYB, and there is no point Y' between $Q^{(k)}$ and B for which there is a component T of H such that $Y'+T+$segment AXB is connected. In the latter case, the argument for obtaining α may be used to obtain an arc $Q^{(k)}B$ lying in $J+I-(H+K)$. The set $AQ^{(k)}+Q^{(k)}B$ contains an arc AB lying in $J+I-(H+K)$; and the theorem is established.

THEOREM 28.* *If the common part of the closed and compact point sets* H *and* K *is a continuum, and if neither* H *nor* K *separates the point* A *from the point* B, *and if furthermore,* $H-H\cdot K$ *and* $K-H\cdot K$ *are mutually separated sets, then* $H+K$ *does not separate* A *from* B.

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* Cf. S. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczno-Fizyczne, vol. 26, 1913. Also Anna M. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

Proof. Suppose on the contrary that $H+K$ separates A from B. Let $M=H+K$ and $T=H\cdot K$. Let S_1 and S_2 denote the components of $S-M$ which contain A and B, respectively. There exist arcs AXB and AYB such that $AXB\cdot H=0$ and $AYB\cdot K=0$. Let X_1 and Y_1 denote the first points of AXB and AYB, respectively, which belong to the boundary of S_2. Thus X_1 and Y_1 belong to $K-T$ and $H-T$, respectively, and hence are not contiguous to each other. Let AX_1 and AY_1 denote intervals of AXB and AYB, respectively. The set AX_1+AY_1 contains an arc $X_1A_1Y_1$ such that $X_1A_1\cdot H=0$ and $A_1Y_1\cdot K=0$. If X_1 is contiguous to any point of S_2, let X_2 denote such a point, and let X_1X_2 denote the arc consisting of these two points. Otherwise, let R denote a connected domain containing X_1 but containing no point or boundary point (other than X_1) of $H+A_1Y_1$, let X_2 denote a point of $R\cdot S_2$, and let X_1X_2 denote an arc lying in R. If Y_1 is contiguous to any point of S_2, let Y_2 denote such a point, and let Y_1Y_2 denote the arc consisting of these two points. Otherwise, let W denote a connected domain containing Y_1 but containing no point or boundary point (other than Y_1) of $K+A_1X_1+X_1X_2$, let Y_2 denote a point of $W\cdot S_2$, and let Y_1Y_2 denote an arc lying in W. Let X_2Y_2 denote either the point X_2 or an arc lying in S_2 according as X_2 is or is not Y_2. It may be readily shown that the set $X_1A_1Y_1+X_1X_2+Y_1Y_2+X_2Y_2$ contains a simple closed curve J such that (1) J contains A_1 and a point B_1 of X_2Y_2, and (2) of the two segments of J from A_1 to B_1, one contains no point of H and the other contains no point of K. Let I denote that complementary domain of J which does not contain T, and let $H_1=H\cdot(J+I)$ and $K_1=K\cdot(J+I)$. It follows from Theorem 27 that there exists an arc A_1B_1 lying in $J+I-(H_1+K_1)$. Thus A_1B_1 contains no point of M. But this is impossible since A_1 and B_1 lie in different complementary domains of M. The theorem is thus established.

THEOREM 29. *If no point of the arc XY separates the point A from the point B, then XY does not separate A from B.*

Proof. Suppose $S-XY=S_A+S_B$, where S_A and S_B are mutually separated sets containing A and B, respectively. Let S_1 denote the set consisting of X together with all points Z, if there are any, such that the interval XZ does not separate A from B, and let $S_2=XY-S_1$. Now S_2 contains Y, and clearly every point of S_1 precedes every point of S_2. Hence there exists either a last point of S_1 or a first point of S_2.

Suppose there exist a point O which is the last point of S_1 and Q which is the first point of S_2. By Theorem 20 the interval OQ does not separate A from B. Hence by Theorem 28, S_1+OQ does not separate A from B. Thus XQ does not separate A from B, contrary to the fact that Q is a point of S_2.

Suppose that there exists a point O which is the last point of S_1. Let $O_1, O_2, O_3, \cdots$ be a sequence of points of S_2 converging to O such that each precedes the next in the order Y to X. By Theorem 28, the interval OO_n separates A from B. Thus, by Theorem 5, Chapter 2, of P.S.T., the point O separates A from B.

Suppose there exists a point O which is the first point of S_2. Let $O_1, O_2, O_3, \cdots$ be a sequence of points of S_1 converging to O such that each precedes the next in the order X to Y. By Theorem 28, the interval O_nO separates A from B. Hence by Theorem 5, Chapter 2, of P.S.T., the point O separates A from B. Again a contradiction is reached and the theorem is established.

The following proposition is false:

Proposition. *If A, B, and C are three distinct points, and if A is not contiguous to either B or C, then there exists a simple closed curve or triune which separates A from $B+C$.*

In Example 1, let A be any point of the triune $T_1T_2T_3$, and let B and C be points of different complementary domains of this triune. There does not exist a simple closed curve or triune which separates A from $B+C$.

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PLANE PEANIAN CONTINUA WITH UNIQUE MAPS ON THE SPHERE AND IN THE PLANE

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Reprinted from the
Transactions of the American Mathematical Society
Vol. 44, No. 1, pp. 58-67
July, 1938

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INTRODUCTION

The plane peanian continuum† M is said to have a unique map on a spherical surface (or a plane) if, and only if, for any topological image M' of M on a sphere (or plane) S' and any topological image M'' of M on a sphere (or plane) S'', every homeomorphism of M' into M'' can be extended to a homeomorphism of S' into S''.

It is the purpose of this paper to characterize the plane peanian continua that have unique maps on the sphere or in the plane.‡

DEFINITIONS. The simple closed curve J of a cyclicly connected continuum C is called a *bounding circuit* of C provided that for any two maximal connected components H and K of $C-J$ the sets $\overline{H}\cdot J$ and $\overline{K}\cdot J$ lie respectively on two distinct arcs AXB and AYB of J.§

A *split circuit* is a bounding circuit J such that $C-J$ contains at least two components.

If C is cyclicly connected, then C is *triply connected* if, and only if, it is impossible to express C as the sum of two closed connected sets A and B such that neither A nor B is an arc and the set $A\cdot B$ consists of two distinct points of C.‖

Theorems IV and V state the principal results of this paper.

THEOREM IV. *The plane peanian continuum M has a unique map on the sphere if, and only if, one of the following conditions holds*:

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* Presented to the Society, September 9, 1937; received by the editors July 22, 1937.

† A plane peanian continuum is a peanian continuum (continuous curve) that has a map (topological image) in the plane or on the sphere. For a characterization of these continua see W. S. Claytor, *Topological immersion of peanian continua in a spherical surface*, Annals of Mathematics, vol. 35 (1934), pp. 809–835. See also Claytor, *Peanian continua not imbeddable in a spherical surface*, ibid., vol. 38 (1937), pp. 631–646.

‡ This problem was suggested by J. R. Kline. The author also wishes to express his appreciation of suggestions by Saunders MacLane which have led to improvements in this paper.

§ For cyclicly connected continua *bounding circuit* is equivalent to *boundary curve* as defined by Claytor, loc. cit., first paper, p. 809. A simple closed curve J of M is called a boundary curve of M provided that there do not exist in $M-J$ distinct components H and K such that (1) a point pair of $\overline{H}\cdot J$ separates a point pair of $\overline{K}\cdot J$ on J, or (2) $\overline{H}\cdot J=\overline{K}\cdot J=$ three distinct points. Both definitions will be found useful in later proofs.

‖ Cf. definition of triply connected graphs, H. Whitney, *Congruent graphs and the connectivity of graphs*, American Journal of Mathematics, vol. 54 (1932), p. 158.

(1) *M is acyclic and consists of either a simple arc or a triod.*[*]

(2) *M contains one cyclic element C*[†] *which is a maximal triply connected cyclic curve of M, and $M-C$ consists of at most a countable number of arcs, $a_1, a_2, a_3, \cdots$, such that $\bar{a}_i \cdot \bar{a}_j = 0$, $(i \neq j)$, and each $\bar{a}_i \cdot C$ is a single point which lies on only one bounding circuit of C,*[‡] *provided that if C is a simple closed curve, then $M-C$ is at most a simple arc.*

THEOREM V. *The plane bounded peanian continuum M has a unique map in the plane if, and only if, M is one of the following curves:*

(1) *a simple arc,*

(2) *a triod,*

(3) *a simple closed curve,*

(4) *a curve M such that M contains a closed 2-cell C and $M-C$ consists of at most a countable number of arcs $a_1, a_2, a_3, \cdots$, such that $\bar{a}_i \cdot \bar{a}_j = 0$, $(i \neq j)$, and each $\bar{a}_i \cdot C$ is a single point which lies on the only bounding circuit of C.*[§]

PRELIMINARY THEOREMS

THEOREM I. *The plane cyclic (cyclicly connected) peanian continuum C is triply connected if, and only if, C contains no split circuit.*

THEOREM II. *The plane cyclic peanian continuum C has a unique map on the sphere if, and only if, C is triply connected.*

COROLLARY 1. *If J is a bounding circuit of C which is not a split circuit, then in every map of C on the sphere (or plane) the image of J is the boundary of a complementary domain of the map of C.*

COROLLARY 2. *If J is a split circuit of C there is some map of C on the sphere (or plane) in which the image of J is the boundary of a complementary domain of the map of C.*[||]

DEFINITION. The point p of M is a *split-point* of M, if and only if, M can be expressed as the sum of two closed connected sets A and B such that $A \cdot B = p$, and such that if A or B is an arc, then p is not an end point of that arc.

THEOREM III. *If the plane peanian continuum M has a unique map on the sphere or in the plane, then M does not contain a split-point.*

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[*] Three arcs PA, PB, PC with P, and only P, common to any two.

[†] G. T. Whyburn, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), p. 168.

[‡] Each point $\bar{a}_i \cdot C$ is a non-local cut-point of C. See Theorem VI and the alternative statement of Theorem IV at the end of this paper.

[§] See Claytor. loc. cit., p. 810, Corollary (C).

[||] This is a generalization of Proposition K, Claytor, loc. cit., first paper, p. 828.

PROOFS OF THEOREMS I–V

Proof of Theorem I. Let C be a map of the given curve on a plane S, and suppose C not triply connected. There exist two points p and q of C such that $C=A+B$ and $A\cdot B=p+q$, where A and B are closed connected sets neither of which is an arc. Let r be a point of $A-p-q$ and s a point of $B-p-q$. Then on any arc rxs lying in $S-p-q$ there is a last point of A from r to s and a first point of B. Let these points be r and s respectively. The open arc $\langle rxs\rangle$ lies in a domain R complementary to C with boundary $J\supset r+s$. Hence this circuit J must pass through p and q. But since neither A nor B is an arc there must exist at least two components of $C-J$; one in $A-A\cdot J$ and one in $B-B\cdot J$. Let N_1 and N_2 be any two components of $C-J$. Then two points of $\overline{N}_1\cdot J$ cannot separate two points of $\overline{N}_2\cdot J$ on J. For N_1 and N_2 are connected sets lying in $S-\overline{R}$ and cannot intersect. If $\overline{N}_1\cdot J=\overline{N}_2\cdot J=$three points, then N_1 and N_2 must intersect; this is impossible. Therefore, since $C-J$ contains at least two components, J is a split circuit.

Now assume that C contains a split circuit J. There is a component N_1 of $C-J$ with limit points all on the arc gxh of J and a component N_2 with limit points all on the arc $gyh=J-\langle gxh\rangle$. The arc gxh may be chosen so that g and h are points of $\overline{N}_1\cdot J$. Now every component of $C-J$ must have its limit points in either gxh or gyh. For suppose some component N of $C-J$ has a limit point d in $\langle gxh\rangle$ and a limit point e in $\langle gyh\rangle$. Every arc of J containing d and e will necessarily contain either g or h, limit points of N_1 which separate d and e on J. But this contradicts the fact that J is a split circuit.

Now let $A=gxh+N_1$ plus all components of $C-J$ (different from N_2) with limit points on gxh, and let $B=gyh+N_2$ plus all components of $C-J$ not included in A. Then $C=A+B$, where A and B are closed connected subsets of C; neither A nor B is a simple arc; and $A\cdot B=g+h$. Therefore C is not triply connected. This completes the proof of Theorem I.

Proof of Theorem II. Suppose C is triply connected. Then every bounding circuit of C has an image which is a c.d.b.* In every map of C. For let C' be any map of C on a sphere S', and J' any bounding circuit of C'. If C' does not consist of a simple closed curve, then there is one, and only one, component of $C'-J'$ (Theorem I), and this component must lie entirely in one of the regions of S' bounded by J'. The other region of S' bounded by J' must be a complementary domain of C'. Obviously every c.d.b. of C' is also a bounding circuit of C.

Therefore if C' is a map of C on a sphere S' and C'' a map of C on S'', every homeomorphism of C' into C'' must preserve complementary domain boundaries and is extendable to S' and S''.* Hence C has a unique map on the sphere.

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* "Complementary domain boundary" will be abbreviated "c.d.b."

Suppose C is not triply connected. Then C must contain a split circuit J (Theorem I). We shall show first that there is a map of C in which the map of J is a c.d.b. Let C be any map of C on a sphere S and suppose J is not a c.d.b. of C. Let R_1 and R_2 be the two regions of S bounded by J. Since J is not a c.d.b. of C there must exist a component N_1 of $C-J$ in R_1 and a component N_2 in R_2. Any two points p and q of $\overline{N}_1 \cdot J$ lie on some c.d.b. of C within $\overline{R}_2$. For if every arc from p to q lying in $\overline{R}_2$ contains a point of C, there would then exist a connected subset of C lying in R_2 with limit points on J that separate p and q.† But such a connected subset would belong to a component of $C-J$, and J would not be a split circuit. Hence p and q lie on some c.d.b. within $\overline{R}_2$. Furthermore any three points p, q, r of $\overline{N}_1 \cdot J$ lie on the same c.d.b. of C within $\overline{R}_2$. To show this suppose p, q, r do not lie on the same c.d.b. within $\overline{R}_2$. Then there are three complementary domains of C in $\overline{R}_2$ each containing a pair of p, q, r on its boundary J_i, $(i=1, 2, 3)$. Let l, m, n be three arcs, one from each of the boundaries J_i such that $l+m+n \supset p+q+r$ and bounds a region R_3 which is a subset of R_2 containing no points of $J_1+J_2+J_3$. This is possible since no two arcs of l, m, n can have a common point. For if two arcs, l, m do have a common point (other than an end point) there would exist a component $N_3 \supset (l+m)$ of $C-J$ such that $\overline{N}_3 \cdot J$ contains p, q, r, and since $\overline{N}_1 \cdot J$ contains p, q, r, the circuit J would not be a split circuit (definition). Now any two points of p, q, r lie together on the same c.d.b. of C within $\overline{R}_3$. For if every arc from p to q (or to r) lying in R_3 contained a point of C, there would exist, as above, a connected subset of C lying in $\overline{R}_3$ and having points on $l+m+n$ that separate p and q on $l+m+n$. There would then exist a component N_4 of $C-J$ such that $\overline{N}_4 \cdot J$ contains p, q, and r; but this is impossible. Now let the circuit J_1 above be the boundary of a domain r_1 complementary to C and lying in R_2. Let l be the arc (of the three l, m, n) which lies on $J_1 \supset p+q$. Then since R_3 contains no points of $J_1+J_2+J_3$, the domain r_1 is a subset of R_2, but not of R_3, and has p and q on its boundary. Let r_2 be a domain complementary to C which is a subset of R_3 and has p and q on its boundary. Now R_3 is not a complementary domain of C since we are assuming that p, q, r

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* V. W. Adkisson, *Cyclicly connected continuous curves whose complementary domain boundaries are homeomorphic, preserving branch points*, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Class 3, vol. 23 (1930), p. 167, Theorem 2. If M is a cyclicly connected continuous curve lying on a sphere S, and T is a continuous (1-1) correspondence such that $T(M)=M$, a necessary and sufficient condition that T be extendable to S is that for every boundary J of a complementary domain of M, $T(J)$ be also the boundary of a complementary domain of M.

† C. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fundamenta Mathematicae, vol. 15 (1930), p. 274, Lemma III′.

do not all lie on the same c.d.b. in $\overline{R}_2$. The above process can then be repeated and a third complementary domain r_3 obtained which is different from r_1 and r_2 but also has p and q on its boundary. This process may be continued indefinitely. Hence there exists an infinite number of domains $r_1, r_2, r_3, \cdots$ each complementary to C and having p and q on its boundary. But this is impossible since there is at most a finite number of complementary domains of C of diameter greater than any $\epsilon>0$,* and the diameter of each r_i is equal to or greater than the distance between p and q. We conclude that there is a c.d.b. L within $\overline{R}_2$ containing p, q, r.

Now any fourth point s of $\overline{N}_1 \cdot J$ must also lie on L. For s and one of the three points p, q, r, say r, must separate the other two, p and q, on J. Since r and s must lie on the same c.d.b. of C in $\overline{R}_2$ there exists an arc $\langle rs\rangle$ in R_2 such that $\langle rs\rangle \cdot C=0$, and in like manner an arc $\langle pq\rangle$ in R_2 such that $\langle pq\rangle \cdot C=0$. But $\langle rs\rangle$ and $\langle pq\rangle$ must then have a common point and hence lie in a common complementary domain of C. Therefore p, q, r, and s all lie on L, and all points of $\overline{N}_1 \cdot J$ must lie on L.

Let D_1 be the complementary domain of C bounded by L, and let (N) represent the set of all components N_i of $C-J$ in R_1 for which $\overline{N}_i \cdot J=\overline{N}_i \cdot L$. Let (N') be a topological image of (N) in D_1 such that $C-(N)+(N')$ is a map of C. This is always possible since obviously it would be possible if C were mapped so that L is a circle. This process can be repeated on the map $C-(N)+(N')$ so that finally a map C' is obtained in which R_1 is a complementary domain of C'. It follows by well known methods in analysis situs that C' is a topological map of C. For if the above process is necessary an infinite number of times,† it involves complementary domains $D_1, D_2, D_3, \cdots$ of C of which only a finite number are of diameter greater than any $\epsilon>0$.

By the same method as above it is possible to map C so that the image of the split circuit J is not a c.d.b. of the image of C since there are at least two components of $C-J$.

Now let C' be a map of C on S' in which the image of J is a c.d.b. of C' and C'' a map on S'' in which the image of J is not a c.d.b. of C''. Then every homeomorphism of C' into C'' is not extendable to S' and S'' since complementary domain boundaries are not preserved. Hence C does not have a unique map, and the theorem is proved.

Corollaries 1 and 2 follow directly from the preceding proof.

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* Schoenflies proved (1908) that the complementary domains of a continuous curve are countable, and that at most a finite number have diameters greater than any $\epsilon>0$. See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 290, 295.

† The components N_i are countable. See R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 360, Theorem 9.

Proof of Theorem III. We shall assume that M has a split point and at the same time a unique map and show that this leads to a contradiction.

Let M be a map on the sphere S, p a split point of M, and A and B two closed connected subsets of M such that $M=A+B$, $A\cdot B=p$, and such that if A or B is an arc, p is not an end point of this arc. Let R be a complementary domain of M whose boundary contains points of both A and B. Let $x\neq p$ be a point of A in the boundary of R and $z\neq p$ a point of B in the boundary of R. Let $\langle xyz\rangle$ be an arc in R, xpz an arc of M, $A\supset$ arc xp, and $B\supset$ arc pz. Let A_1 be a connected component of $A-xp$ such that $\overline{A}_1$ has a point r on $xp-x$, and B_1 a connected component of $B-pz$ such that $\overline{B}_1$ has a point s on $pz-z$. Since neither A nor B is a simple arc with end point at p, it is possible to choose x and z so that this latter condition may be satisfied. There are two cases to consider.

Case 1. $r\neq s$. Let M' represent a map of M on the sphere S'. Let T be a homeomorphism such that $T(M)=M'$ and $T(S)=S'$. This is possible since we assume that M has a unique map. Throughout this proof a primed set will indicate the topological image under T of the unprimed set. Now either the set A_1 lies in the region D_1 of S bounded by the simple closed curve $C=xpzyx$ while B_1 lies in the other region D_2 bounded by C, or A_1 and B_1 lie in the same region, say D_1. We shall assume the latter, but the proof is practically the same in either case.

Since T is extendable to S and S', the sets A_1' and B_1' lie in D_1'. We now construct a new map M'' of M on S' as follows: Let H_i', $(i=1,2)$, be the subset of $A'-$arc $x'p'$ that lies on D_i'. The set H_1' includes A_1'. Let H_1'' be a topological image of H_1' lying in $D_2'-B'\cdot D_2'$, and H_2'' a topological image of H_2' lying in $D_1'-B'\cdot D_1'$ such that $x'p'+H_1''+H_2''$ is a topological image of A'. Then $M''=x'p'+H_1''+H_2''+B'$. Let U be a homeomorphism such that $U(M)=M''$ and $U(S)=S'$. Let $U(xyz)=x'y''z'$, $U(H_i)=H_i''$, and for points in $B+xpz$ let $U=T$. Since U is extendable, $U(D_1)=D_1''$ is a region of S' containing $U(M\cdot D_1)$. Let f be any arc of S' from a point of A_1'' (the topological image of A_1' in H_1'') to a point of B_1' which lies entirely in D_1'' including end points. Such an arc must intersect C', the boundary of D_1', since B_1' lies in D_1' and A_1'' lies outside D_1'; and since $x'p'z'$ is in the boundary of D_1'' the arc f must intersect $\langle x'y'z'\rangle$. Then any arc g from r' to s' lying, except for end points, in D_1'' must intersect $\langle x'y'z'\rangle$. To show this let d and e be regions about r' and s' respectively, that do not contain points of $x'y'z'$. It is possible to obtain an arc t that lies entirely in d and joins a point of $g-r'$ to a point of A_1''. In like manner we obtain an arc u in e joining a point of $g-s'$ to a point of B_1'. The arcs t and u plus the proper subset of g then yield an arc h from A_1'' to B_1' lying entirely in D_1''. But h must then intersect $\langle x'y'z'\rangle$, and since neither t nor u can intersect $\langle x'y'z'\rangle$ the arc g must have a point in common with $\langle x'y'z'\rangle$. Hence there must be a connected subset of $x'y'z'$ lying in D_1'' with two points on $C''=x'y''z'p'x'$ (the boundary of D_1'') that separate r' and s' on C''.* One of these points must obviously lie on the arc $r's'$ of C'' where $r's'\subset\langle x'p'z'\rangle$. But this is impossible since $\langle x'p'z'\rangle$ and $\langle x'y'z'\rangle$ have no common points. Therefore the assumption that every homeomorphism of M is extendable has led to a contradiction, and we conclude that M has no unique map.

Case 2. $r=s=p$. We use the same notation as in Case 1 and obtain in the same manner the map M''. Any arc joining a point of A_1'' to a point of B_1' and lying entirely in D_1'' must intersect $\langle x'y'z'\rangle$. Let d be a region about p' that contains no points of $x'y'z'$. Let f be an arc from A_1'' to B_1' lying in D_1'' and also in d. Then f cannot intersect $\langle x'y'z'\rangle$ but must intersect $\langle x'y'z'\rangle$. This contradiction shows that M has no unique map and completes the proof of Theorem III.

Proof of Theorem IV. First, assume that M has a unique map. If M is acyclic it must consist of either a simple continuous arc or a single triod since M contains no split point (Theorem III). Obviously these are the only two acyclic peanian continua without split-points.

If M is not acyclic it contains at least one cyclic element C which is a maximal cyclic curve of M. Since M cannot contain a split-point there is only one such cyclic element C. For if there were a second cyclic element C_1 which is a maximal cyclic curve of M, there would exist a cut-point p of M separating C and C_1,† and obviously p would also be a split-point.

If $M-C$ contains a maximal connected acyclic subset with a branch-point p, or if $M-C$ contains two arcs with a common end point p on C, then p is in either case a split-point. Therefore, $M-C$ contains at most a countable number of simple arcs‡ with distinct end points on C.

We shall now assume that C is not triply connected and obtain a contradiction of the assumption that M has a unique map. If C is not triply connected, C contains a split circuit J (Theorem I). Let M be a map of M on the sphere S and assume that the components N_1 and N_2 of $C-J$ lie in the regions R_1 and R_2 of S bounded by J. There exists a c.d.b. L of C lying in $\overline{R}_2$ such that $\overline{N}_1\cdot J=\overline{N}_1\cdot L$ (this was shown in the proof of Theorem II). Let D be the complementary domain of C bounded by L. Let $\sum a_i$ be the set of arcs of M lying in D with end points in $\overline{N}_1\cdot L$ and $\sum b_i$ the set of arcs of $M\cdot D$ not included in $\sum a_i$. We now obtain a new map M' of M on S as follows: Let

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* Kuratowski, loc. cit., Lemma III′.

† See G. T. Whyburn, loc. cit., p. 168.

‡ R. L. Wilder, loc. cit., p. 360, Theorem 9.

(K) represent all arcs of $M-C$ with end points on N_1, and let N_1' be a map in D of $N_1+(K)$ such that $\overline{N}_1 \cdot L = \overline{N}_1' \cdot L$ and $C-N_1+N_1'$ is a topological image of $C+(K)$. Let D_1 be the complementary domain of $M-[N_1+(K)]$ that contains $N_1+(K)$, and let $\sum a_i'$ be a topological image of $\sum a_i$ in D_1 such that each arc $\bar{a}_i'$ has one end point on L, and the set of all these end points of $\sum \bar{a}_i'$ is identical with the set of end points of $\sum \bar{a}_i$ on L. For each b_i we obtain a new arc b_i' as follows: If $\bar{b}_i \cdot N_1' = 0$, then $b_i' = b_i$. If $\bar{b}_i \cdot N_1' \neq 0$, let d_i be a region about Q_i (the end point of b_i on L) containing no point of N_1'. Then b_i' is taken as a simple arc which is a subset of b_i and lies in d_i with end point Q_i. Then $M' = M - N_1 - (K) + N_1' - \sum a_i + \sum a_i' - \sum b_i + \sum b_i'$ is a topological image of M. But every homeomorphism of M into M' cannot be extended to S since L is a c.d.b. of M but not a c.d.b. of M'. Hence M does not have a unique map, and this contradiction leads to the conclusion that C must be triply connected.

Now suppose C does not consist of a simple closed curve, and there is an arc b in $M-C$ with end point p on two bounding circuits, J and L, of C. Let $(M-b)$ be any map of $(M-b)$ on S. Then J and L bound two complementary domains of C and are outer boundaries of two complementary domains D_1 and D_2 of $M-b$. Let M' be a map of M obtained by mapping b in D_1 with end point at p, and M'' a map of M obtained by mapping b in D_2 with end point at p. We have now two essentially different maps of M. Hence the end points of the arcs in $M-C$ that lie in C must lie in one, and only one, bounding circuit of C.

If C consists of a simple closed curve, $M-C$ cannot contain more than one arc. The proof of this statement is not difficult and will be omitted.

If M consists of a simple continuous arc, the sufficiency of condition (1) follows from the fact that any homeomorphism between two arcs is extendable to a homeomorphism of their planes.* If M consists of a simple closed curve plus an arc, the proof that M has a unique map is easily obtained from the Schoenflies theorem that any homeomorphism between two simple closed curves can be extended to a homeomorphism of their planes.

The proof that a triod has a unique map may also be obtained by a simple application of the Schoenflies theorem.

Suppose C is not a simple closed curve. Let M be a map of M on S, and M' a map on S'. The map of C is unique (Theorem II), and from Corollary 1 we see that a c.d.b. of C in one map has an image which is a c.d.b. in every map of C. Let $\sum a_i$ be the arcs of $M-C$ with end points on the same bounding circuit J of C. Then in the map of C on S these arcs lie in the same complementary domain D of C bounded by J, and the map $\sum a_i'$ of $\sum a_i$ on S' must lie in the complementary domain D' of C' bounded by J' (the image of J) since the end points of $\sum \bar{a}_i'$ lie on J' and on no other bounding circuit of C'.

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* R. L. Moore, *Conditions under which one of two given closed linear point sets may be thrown into the other one by a continuous transformation of a plane into itself*, American Journal of Mathematics, vol. 48 (1926), p. 67.

Now any homeomorphism $T(M)=M'$ can be extended to D and D'. Let pq be one of the arcs $\bar{a}_i$ in $\bar{D}$ with end point p on J such that either pq or its image $p'q'$ on S' is of diameter greater than some $\epsilon>0$. Let $\langle qxy\rangle$ be an arc in $D-\sum a_i$ where y is a point of J but not an end point of some arc a_i in D. Let $\langle q'x'y'\rangle$ be an arc in $D'-\sum a_i'$ where $y'=T(y)$. Let r and s be points of J that separate p and y, and let $r'=T(r)$ and $s'=T(s)$. Let a_k be any arc of $\sum a_i$ that lies in the subset d of D bounded by $pqxyrp$. Then a_k' must lie in the subset d' of D' bounded by $p'q'x'y'r'p'$. For if a_k' lies in $D'-\bar{d}'$, it must have an end point at either p' or y'. But this is impossible since y was selected not to be an end point, and p cannot be an end point common to two arcs in D. Hence if $\sum b_i$ is the subset of $\sum a_i$ lying in d, then $T(\sum b_i)$ is the subset of $\sum a_i'$ lying in d'. The same would be true of any similar subdivision of D and D'. In fact *sides are preserved under* T as used by Gehman.[*] Therefore T can be extended to D and D', and in like manner to each complementary domain of C and C', and finally to S and S'. The proof would necessarily be a partial duplication of Gehman's proof and is omitted. This completes the proof of Theorem IV.

Proof of Theorem V. Cases (1), (2), (3) of Theorem V follow easily from Theorem IV.

If M has a unique map and is not acyclic or not a simple closed curve, Theorem III shows that M contains one, and only one, cyclic element C which is a maximal connected cyclic curve of M. Furthermore C cannot contain two distinct bounding circuits or consist of a single circuit which would be the outer boundary of two complementary domains of M in any map. For if this were true there would be a circuit J which would be the outer boundary of a bounded complementary domain R of M in some map on a plane S (Corollary 1). Now let M' be a map of M on a plane S' in which the image of the boundary of R is the boundary of the unbounded complementary domain R' of M'. Then obviously any homeomorphism of M into M' which carries the boundary of R into the boundary of R' cannot be extended to R and R'. Therefore C contains only one bounding circuit, and Claytor's result (Corollary (C), p. 810) shows that C consists of a closed 2-cell.

Since M cannot contain a split point, $M-C$ can consist of at most a countable number of distinct arcs with distinct end points on the bounding circuit of C.

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[*] H. M. Gehman, *On extending a continuous* (1-1) *correspondence of two plane continuous curves to a correspondence of their planes*, these Transactions, vol. 28 (1926), proof of Theorem I, pp. 256–260.

Now suppose M consists of the curve (4) in the theorem. The arcs $M-C$ must all lie in the unbounded complementary domain of C in any map of M in the plane. Let D and D' be the two unbounded complementary domains of two maps of M and M' respectively. Then any homeomorphism of M into M' can be extended to D and D' as indicated in the last paragraph of the proof of Theorem IV.

A point p of a peanian continuum M is said to be a *local cut-point* of M if, and only if, p is a cut-point of some connected open subset of M.*

The following theorem will be stated without proof:

THEOREM VI. *If M is a peanian continuum in a plane S, any non-cut-point p of M lies on two (or more) complementary domain boundaries of M if, and only if, p is a local cut-point of M.*†

Now if M is a plane peanian continuum and C a maximal cyclic curve of M, every point in $C\cdot(M-C)$ must lie on a bounding circuit of C. Since C contains no cut-point (of C) we can make the following alternative statement of Theorem IV:

The plane peanian continuum M has a unique map on the sphere if, and only if, one of the following conditions holds:

(1) *M is acyclic and consists of either a simple arc or a triod,*

(2) *M contains one cyclic element C which is a maximal triply connected cyclic curve of M, and $M-C$ consists of at most a countable number of arcs, $a_1, a_2, a_3, \cdots$, such that $\bar{a}_i\cdot\bar{a}_j=0$, $(i\neq j)$, and each $\bar{a}_i\cdot C$ is a non-local separating point of C, provided that if C is a simple closed curve, then $M-C$ is at most a simple arc.*

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* See G. T. Whyburn, *Local separating points of continua*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 305–314.

† This theorem is well known to topologists although the author has been unable to find it stated explicitly in any published paper. See, however, G. T. Whyburn, *Local separating points of continua*, loc. cit., Theorem 6, and G. T. Whyburn, *Concerning points of continuous curves defined by certain im kleinen properties*, Mathematische Annalen, vol. 102 (1929), pp. 313–336, Theorem 31.

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FIXED POINTS UNDER TRANSFORMATIONS OF CONTINUA WHICH ARE NOT CONNECTED IM KLEINEN

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Reprinted from the
Transactions of the American Mathematical Society
Vol. 44, No. 1, pp. 18-24
July, 1938

FIXED POINTS UNDER TRANSFORMATIONS OF CONTINUA WHICH ARE NOT CONNECTED IM KLEINEN[*]

BY

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1. Introduction. Ayres[†] has shown that if M is a compact continuous curve in the plane which does not separate the plane and T is a reversibly continuous transformation of M into a subset of itself, then T leaves some point of M invariant. He also proves similar theorems for special cases of non-planar continuous curves. The purpose of this paper is to extend these results to certain types of continua which are not connected im kleinen.

2. Preliminary theorem and lemmas. We prove first the following preliminary theorems and lemmas.

THEOREM I. *If M is a compact continuum in a metric space but is not an indecomposable continuum, and if T is any reversibly continuous transformation of M into a subset of itself, then some proper subcontinuum N of M contains a point of $T(N)$, the image of N under T.*

Proof. If T carries M into a proper subset of itself, then $T(M)$ contains its image $T^2(M)$ and the theorem is true. Suppose then that T carries M into itself. It can easily be shown that if M is not indecomposable, it does not contain two mutually exclusive composants. The transformation T carries a composant of M into a composant of M. It has been shown that a composant[‡] of a continuum is the sum of a countable number of continua $N_1, N_2, N_3, \cdots$, where for each integer i, N_i contains N_{i-1}.

Let V be any composant of M, and let $T(V)$ be its image under T. Then V and $T(V)$, by what was said above, must have a point in common. Let V be expressed as $\sum_{i=1}^{\infty} V_i$, where for each integer i, V_i is a continuum which contains V_{i-1}. Then $T(V) = \sum_{i=1}^{\infty} T(V_i)$ where for each integer i, $T(V_i)$ is the image of V_i under T and is a continuum which contains $T(V_{i-1})$. For some integer j, V_j contains a point of $T(V_j)$, for if we suppose the contrary to be true, it is obvious that V contains no point of $T(V)$, contradicting the fact that V and $T(V)$ have a point in common. V_j is therefore a continuum satisfying the conclusions of the theorem.

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[*] Presented to the Society, September 10, 1937; received by the editors May 29, 1937.

[†] W. L. Ayres, *Some generalizations of the Scherrer fixed-point theorem*, Fundamenta Mathematicae, vol. 16 (1930), pp. 333–336.

[‡] See R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, p. 75.

LEMMA 1. *If M is a compact continuum in a metric space and M is not an indecomposable continuum and contains no continuum which is the sum of two continua whose common part is disconnected, and if T is a reversibly continuous transformation of M into a subset of itself and neither T nor T^{-1} carries a proper subcontinuum of M into itself, then there is a proper subcontinuum L_0 of M such that L_0 contains a point of $T(L_0)$ but such that for $n>1$, the common part of L_0 and $T^n(L_0)$ is vacuous.*

Proof. By Theorem I, some proper subcontinuum K_0 of M contains a point of its image $T(K_0)$. For each positive or negative integer n, let K_n designate $T^n(K_0)$. By a hypothesis of the theorem, K_0 is not identical with K_1, for suppose $K_0=K_1$. Then K_0 is a proper subcontinuum of M which is carried into itself by T. Furthermore $K_0\cdot K_1$ is not identical with K_0 or with K_1; for suppose $K_0\cdot K_1$ is identical with K_0. Then T carries K_0 into a subset of itself. Consider the continuum $A=\prod_{n=0}^{\infty}K_n$. Then $T(A)=A$, and T carries A, a proper subcontinuum of M, into itself. Similarly, if $K_0\cdot K_1$ is identical with K_1, T^{-1} carries a proper subcontinuum of M into itself; thus we have a contradiction of a hypothesis of the theorem.

There exists an integer r such that the $\prod_{i=0}^{r}K_i$ is not vacuous but such that $\prod_{i=0}^{r+1}K_i$ is vacuous. This can be shown as follows. $K_0\cdot K_1$ is not vacuous. Suppose $\prod_{i=0}^{s}K_i$ contains a point for each integer s. Then $B=\prod_{n=0}^{\infty}K_n$ is not vacuous and is a proper subcontinuum of M which is carried into itself by T, which is, again, a contradiction of a hypothesis of the theorem. Let L_0 be the continuum $\prod_{i=0}^{r-1}K_i$, where r is an integer such that $\prod_{i=0}^{r}K_i$ is not vacuous but $\prod_{i=0}^{r+1}K_i$ is vacuous. For each integer i, let $L_i=T^i(L_0)$. Then L_1 is $\prod_{i=1}^{r}K_i$. $L_0\cdot L_1$ is not vacuous since $L_0\cdot L_1=\prod_{i=0}^{r}K_i$ which is not vacuous. $L_0\cdot L_2$ is vacuous. For suppose L_0 contains a point of L_2. Then $\prod_{i=0}^{r-1}K_i$ contains a point P of $\prod_{i=2}^{r+1}K_i$. It follows that P is in $\prod_{i=0}^{r+1}K_i$, which by hypothesis is vacuous. Furthermore L_0 contains no point of L_r, $r>2$. For suppose the contrary and that s is the smallest integer greater than two such that L_0 contains a point of L_s. Consider the two continua, L_0+L_s and $L_1+L_2+\cdots+L_{s-1}$. Their common part is the sum of the two mutually exclusive continua, $L_0\cdot L_1$ and $L_{s-1}\cdot L_s$. Then $\sum_{i=0}^{s}L_i$ is a subcontinuum of M which is the sum of two continua whose common part is disconnected. This is a contradiction of a hypothesis of the theorem. The lemma is therefore true.

LEMMA 2. *If, in a metric space, M is a compact non-degenerate continuum but is not an indecomposable continuum and does not contain a continuum which is the sum of two continua whose common part is disconnected, and if T is* *a reversibly continuous transformation of M into a subset of itself, then T carries some proper subcontinuum of M into itself.*

Proof. Suppose T does not carry a proper subcontinuum of M into itself. Then by Lemma 1, there exists a sequence of subcontinua L_0, L_1, L_2, $\cdots$ such that (1) for each positive and negative integer n, L_n is $T^n(L_0)$, (2) L_n and $L_{n\pm 1}$ have a point in common, but L_n and L_{n+r}, $|r|>1$, do not have a point in common. Let V_0 be an irreducible continuum from $L_{-1}\cdot L_0$ to $L_0\cdot L_1$ which is a subcontinuum of L_0. Let V_n, for each positive or negative integer, designate $T^n(V_0)$. Then since T is reversibly continuous, it can be easily shown that V_n is irreducible from $L_{n-1}\cdot L_n$ to $L_n\cdot L_{n+1}$. Let P_0 be a point of V_0 not in $L_{-1}\cdot L_0$ or $L_0\cdot L_1$. There is such a point for otherwise L_{-1} would contain a point of L_1. For each positive or negative integer n, let P_n designate $T^n(P_0)$. Let M_0 be an irreducible continuum from P_0 to P_1 which is a subset of $V_0+V_1+L_0\cdot L_1$. Then since T is reversibly continuous, it follows that (1) M_n for each integer n is irreducible from P_n to P_{n+1}, (2) $M_n\cdot M_{n\pm 1}$ is not vacuous, but $M_n\cdot M_{n+r}$, $|r|>1$, is vacuous, (3) $(M_i+M_{i-1})\cdot V_i$ is an irreducible continuum from $L_{i-1}\cdot L_i$ to $L_i\cdot L_{i+1}$.

Since M is not indecomposable, M is the sum of two proper subcontinua H and K. One of these continua contains points of infinitely many of the continua M_0, M_1, M_2, $\cdots$. Suppose H contains points of infinitely many of these continua. Let j be the smallest integer greater than or equal to zero such that H contains a point of M_j. Let r be any integer greater than 5 such that H contains a point of M_{j+r}. Let H' designate the common part of H and the continuum $\sum_{i=j}^{j+r} M_i$. But H' is itself a continuum since the common part of two subcontinua of M cannot be disconnected. H' must contain points of $L_{j+1}\cdot L_{j+2}\cdot M_{j+1}$, $L_{j+2}\cdot L_{j+3}\cdot M_{j+2}$, $\cdots$, $L_{j+r-2}\cdot L_{j+r-1}\cdot M_{j+r-2}$, since each of these continua separates M_j from M_{j+r} in $\sum_{i=j}^{j+r} M_i$. It follows that H' contains P_{j+2}, P_{j+3}, $\cdots$, P_{j+r-2}, since for each integer s, P_{j+s} lies in a subcontinuum of $\sum_{i=j}^{j+r+1} L_i$ which is irreducible from $L_{j+s-1}\cdot L_{j+s}$ to $L_{j+s}\cdot L_{j+s+1}$. Then H' contains each of the continua M_{j+2}, M_{j+3}, $\cdots$, M_{j+r-3}; since each of these continua is irreducible between two points of $\sum_{i=j+1}^{j+r-2} P_i$. Since r can be taken arbitrarily large, it follows that H' contains M_{j+n} for each integer n greater than 1.

Consider the continuum $C=\overline{\sum_{i=j+2}^{\infty} M_i}$. It is obvious that $T(C)$ is a subset of C. It follows that C is identical with M, for otherwise C contains a proper subcontinuum of M which is carried into itself by T, contrary to hypothesis. But by the argument given above C is a subset of H; therefore H is identical with M. This contradicts the assumption that H is a proper subcontinuum of M. The lemma is therefore true.

3. A general theorem on fixed points.

We may now prove the general theorem:

THEOREM II. *If M is a compact continuum in a metric space, and if M does not contain an indecomposable continuum and does not contain a continuum which is the sum of two continua whose common part is disconnected, then every reversibly continuous transformation of M into a subset of itself leaves some point of M invariant.*

Proof. By Lemma 2, T carries some proper subcontinuum M_1 of M into itself. Let $M_1, M_2, \cdots, M_\omega, M_{\omega+1}, \cdots$ be a well ordered sequence β of subcontinua of M having the following properties: (1) for each ordinal number θ which has an immediate predecessor, M_θ is a proper subcontinuum of $M_{\theta-1}$ which is carried into itself by T; (2) for each ordinal number ϕ which has no immediate predecessor, M_ϕ is a proper subcontinuum of M_α which is carried into itself by T, where α is any ordinal which is less than ϕ; (3) if M_λ is any non-degenerate continuum which belongs to the sequence β, $M_{\lambda+1}$ is a proper subcontinuum of M_λ belonging to the sequence β. Since M is metric, it is completely separable, and it follows that the sequence β thus defined is countable. Therefore there exists a countable simple subsequence $M_{n1}, M_{n2}, \cdots$ running through β. That is, if M_λ is any element of β, there is an integer i such that M_{ni} is a subcontinuum of M_λ. Since M is compact, the point set $P=\prod_{i=1}^{\infty} M_{ni}$ is a compact continuum and is carried into itself by T, since each of the continua M_{ni} is carried into itself by T. But P cannot be a non-degenerate continuum, for if it is, it has a proper subcontinuum which belongs to the sequence β; and this contradicts the definitions of P and of the sequences β and M_{ni}. It follows that P is a point of M, invariant under T.

4. Applications to continua in the plane.

We have now the following applications:

THEOREM III. *If M is a compact continuum in the plane which contains no indecomposable continuum, which does not separate the plane, and which contains no domain, then every reversibly continuous transformation of M into a subset of itself leaves some point invariant.*

Proof. It follows from a theorem proved by S. Janiszewski* and also by Miss Mullikan† that a sufficient condition that a compact continuum M separate the plane is that M be the sum of two continua whose common part is disconnected. Then if M does not separate the plane and contains no domain, no subcontinuum of M separates the plane, and the hypotheses of Theorem II are satisfied. The conclusions of Theorem III then follow.

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* S. Janiszewski, *Sur les coupures du plan faites par les continus*, Prace Matematyczno-fizyczne, vol. 26 (1913), pp. 11–63.

† Anna M. Mullikan, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

THEOREM IV. *If D is a bounded simply connected domain in the plane which, together with its boundary, does not separate the plane and whose outer boundary M contains no indecomposable continuum, then every reversibly continuous transformation of $\bar{D}$ into itself leaves some point of $\bar{D}$ invariant.*

Proof. Without loss of generality we may suppose the boundary of D to be identical with its outer boundary, for the compact complementary domain of the outer boundary M of D is itself a simply connected domain which is a subset of $\bar{D}$ and whose boundary is M. Carathéodory* has shown that there exists a conformal and reversibly continuous transformation T_1 of D into the interior I of a given circle J, that there exists a reversibly one-to-one correspondence, also designated by T_1, between the prime ends of D and the points of J, and that this correspondence is characterized as follows: If $P_1, P_2, P_3, \cdots$ is a sequence of points of D converging to a prime end E_P of D (using Carathéodory's definition of convergence), then the sequence of points $T_1(P_1), T_1(P_2), T_1(P_3), \cdots$ converges (in the usual sense) to the point P of J with which E_P is associated by this correspondence; and conversely, if $Q_1, Q_2, Q_3, \cdots$ is a sequence of points of I converging to a point Q of J, then the sequence of points $T_1^{-1}(Q_1), T_1^{-1}(Q_2), T_1^{-1}(Q_3), \cdots$ converges to the prime end E_Q with which Q is associated by the correspondence T_1. Let T be any reversibly continuous transformation of $\bar{D}$ into itself. It can be easily shown that T carries a prime end of D into a prime end of D. Let T_2 be a transformation of the continuum $J+I$ into itself defined as follows: If P is any point of I, let $T_2(P)$ be the point $T_1[T(T_1^{-1}(P))]$. If P is any point of J, let $P_1, P_2, \cdots$ be a sequence of points of I converging to P. Let E_P be the prime end of D associated with P by T_1. Then the sequence $T_1^{-1}(P_1), T_1^{-1}(P_2), T_1^{-1}(P_3), \cdots$ converges to E_P. Since T is reversibly continuous, the sequence $T(T_1^{-1}(P_1)), T(T_1^{-1}(P_2)), \cdots$ converges to the prime end $T(E_P)$ into which E_P is transformed by T. It follows that the sequence of points of I, $T_1[T(T_1^{-1}(P_1))], T_1[T(T_1^{-1}(P_2))], \cdots$ converges to a point of J. That point is defined as $T_2(P)$. Since $T_1[T(T_1^{-1}(P_i))]$ is defined as $T_2(P_i)$, and since T is reversibly continuous on $\bar{D}$, and T_1 is a reversible continuous transformation of D into I, it follows that T_2 is reversibly continuous on I. Also from the definition of T_2 it follows that if $P_1, P_2, P_3, \cdots$ is a sequence of points of I converging to a point P of J, then the sequence $T_2(P_1), T_2(P_2), T_2(P_3), \cdots$

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* C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete,* Mathematische Annalen, vol. 73 (1912), pp. 323–370.

converges to $T_2(P)$, and the sequence $T_2^{-1}(P_1)$, $T_2^{-1}(P_2)$, $T_2^{-1}(P_3)$, $\cdots$ converges to $T_2^{-1}(P)$. Suppose that Q_1, Q_2, Q_3, $\cdots$ is a sequence of points of J converging to a point Q of J. For each positive integer n, let Z_n be a point of I such that each of the distances $d(T_n, Q_n)$ and $d[T_2(Z_n), T_2(Q_n)]$ is less than $1/n$. That such a point Z_n exists is shown as follows: There exists a sequence of points Z_{n1}, Z_{n2}, Z_{n3}, $\cdots$ converging to Q_n. Then by the discussion above, the sequence $T_2(Z_{n1})$, $T_2(Z_{n2})$, $T_2(Z_{n3})$, $\cdots$ converges to $T_2(Q_n)$. There exists an integer i such that $d(Z_{ni}, Q_n) < 1/n$ and $d[T_2(Z_{ni}), T_2(Q_n)] < 1/n$. Then Z_{ni} has the property required of Z_n. We see then that the convergence of the sequence Q_1, Q_2, $\cdots$ to Q implies the convergence of the sequence Z_1, Z_2, $\cdots$ to Q. This in turn implies the convergence of $T_2(Z_1)$, $T_2(Z_2)$, $\cdots$ to $T_2(Q)$ and therefore the convergence of $T_2(Q_1)$, $T_2(Q_2)$, $\cdots$ to $T_2(Q)$. Similarly the sequence of points $T_2^{-1}(Q_1)$, $T_2^{-1}(Q_2)$, $\cdots$ converges to $T_2^{-1}(Q)$, and T_2 is a reversibly continuous transformation of the continuum $J+I$ into itself. By a fundamental theorem on fixed points, T_2 leaves some point of $J+I$ fixed. If T_2 leaves a point of I fixed, then obviously T leaves a point of D fixed. If T_2 leaves a point P of J fixed, then T carries some prime end E_P of D into itself. Let N_P be the continuum in the boundary of D associated with the prime end E_P in the sense that every sequence of points of D converging to E_P has a subsequence which converges in the usual sense to a point of N_P. Then T carries N_P into itself. N. E. Rutt* has shown that a necessary condition that such a continuum as N_P shall be the whole boundary of D is that the boundary of D shall be indecomposable or the sum of two indecomposable subcontinua. Since by hypothesis the boundary of D contains no indecomposable continuum, N_P is a proper subcontinuum of the boundary of D. N_P then is a compact continuum which does not separate the plane and which contains no domain. It satisfies the hypothesis of Theorem III; therefore T leaves some point of N_P fixed. Thus, in any case, T leaves a point of $\bar{D}$ invariant.

5. **An example.** There exist continua which admit of no continuous transformation into themselves except the identity. We give below an example of a compact acyclic continuous curve in the plane having this property.

Let A_1 be any arc in the plane of length 1. Let A_2 be the sum of A_1 and two arcs of length $1/2^2$ having no point in common with each other or with A_1 except that each has one end point at the midpoint of A_1. For each integer n, let E_{n-1} designate the set of points of A_{n-1} of Menger order greater than two. Let A_n be the sum of A_{n-1} and n arcs of length $1/2^n$ having no point in common with each other or with A_{n-1} except that they all have in common one end point which is the midpoint of a component of $A_{n-1}-E_{n-1}$ whose length is not less than the length of any other component of $A_{n-1}-E_{n-1}$. Let $M=\sum_{r=1}^{\infty}A_r$. Then M is obviously an acyclic continuous curve. Let K be the set of points of M of Menger order greater than two. From the definition of M it follows that no two points of K are of the same Menger order and that K is everywhere dense in M. Let T be any continuous transformation of M into itself. T carries a point of given Menger order into a point of the same Menger order and therefore leaves each point of K fixed. Since K is everywhere dense in M, T leaves each point of M fixed.

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* N. E. Rutt, *Prime ends and indecomposability*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 265–273.

It should be noted that M is homeomorphic with a proper subset of itself. E. W. Miller* has given an example of an acyclic continuous curve which is not homeomorphic with any proper subset of itself, but which however does not have the property of the example we have given, since his acyclic continuous curve contains an arc which contains no points of Menger order greater than two. I have not been able to find an example of an acyclic continuous curve in the plane whose only transformation into a subset of itself is the identity.

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* E. W. Miller, *The Zarankiewicz problem*, Bulletin of the American Mathematical Society, vol. 38 (1932), pp. 831–834.

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CONCERNING CERTAIN TOPOLOGICALLY FLAT SPACES†

BY

F. BURTON JONES

In any space satisfying Axioms 0, 1, 2, 3, 4, and 5 of R. L. Moore's *Foundations of Point Set Theory*‡ a large body of Plane Analysis Situs theorems holds true. Nevertheless, Moore has an example§ of a completely separable space satisfying these axioms which because of its "hilly" character is not homeomorphic with a subset of a plane or a sphere. In this paper Axiom 5 is replaced by an axiom, Axiom 5_1^*, which is not satisfied by spaces of this "hilly" character. As a result, any completely separable (or metric) space satisfying Axioms 0–4 and 5_1^* is homeomorphic with a subset of a plane or a sphere. However, in the presence of Axioms 0–4 neither of the Axioms 5 and 5_1^* follows from the other.

Axiom 5_2|| of *Foundations* suggests the following two axioms:

AXIOM 5_1^1. *If P is a point of region R, there exists in R a domain D containing P such that the boundary of D is compact.*¶

AXIOM 5_1^2. *If P is a point of a region R, there exists in R a domain D containing P such that the boundary of D is connected.*††

There is, however, a certain amount of similarity between these last two axioms. For suppose that a space satisfies Moore's Axioms 0, 1, and 2 and

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† Parts of this paper were presented to the Society on October 28, 1933, June 20, 1934, and February 23, 1935; received by the editors July 27, 1936. The author wishes to acknowledge his indebtedness and appreciation to Professor R. L. Moore for much encouragement and many helpful suggestions during this investigation and for having stimulated his interest in mathematics.

‡ American Mathematical Society Colloquium Publications, volume XIII, New York, 1932. Hereafter, in this paper, this book will be referred to as *Foundations*.

§ This example of Moore's, although unpublished, has been well known among his students for some four or five years.

|| If P is a point of a region R, there exists a connected domain D containing P and bounded by a compact continuum T such that $D+T$ is a subset of R.

¶ A space satisfying Axiom 5_1^1 is said to be *locally peripherally compact.*

†† A space satisfying Axiom 5_1^2 is said to be *locally peripherally connected.*

Axiom 5_1^1.[†] Then if P is a point of a region R, there exists a region R' containing P and lying together with its boundary in R. And by Axiom 5_1^1 there exists in R' a domain D' containing P whose boundary β is compact. But β may be covered by a finite collection $D_1, D_2, D_3, \cdots, D_j$ of connected domains such that for each n, $n \leqq j$, $\overline{D}_n$ is a subset of $R-P$.[‡] Let D denote the component of $R-\sum \overline{D}_n$ which contains P. Thus the following theorem holds true and will be used hereafter as

AXIOM 5_1^*. *If P is a point of a region R, there exists in R a domain D containing P whose boundary is a subset of the sum of a finite number of continua lying in $R-D$.*

In a sense Axiom 5_1^* is common to both Axioms 5_1^1 and 5_1^2 in that it follows as a theorem from either of them in the presence of Axioms 0, 1, and 2.

Now let S denote a space satisfying Moore's Axioms 0, 1, and 2.

THEOREM A. *If Axiom 5_1^* holds true at each of two distinct points A and B of S and no point of S separates A from B, then there exists in S a simple closed curve J containing A and B.*[§]

Proof. Let AB denote an arc from A to B.[‖] If R is a region containing A but not B, suppose that there exists no point O of $AB \cdot R$ such that no point X of the interval AO of AB separates A from O in R. Then there exists a sequence α of points of AB which converge to A such that if X and Y are points of α and X follows Y in α, then X separates A from Y in R. Hence, for each point X of α except the first, $R-X$ is the sum of two domains U_X and V_X such that V_X contains A and U_X contains every point of α preceding X in α. But no point separates A from B; so for each point X of α, there exists

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[†] Axioms referred to are stated either in the text of this paper or in *Foundations.*

AXIOM 0. *Every region is a point set.*

AXIOM 1. *There exists a sequence $G_1, G_2, G_3, \cdots$ such that* (1) *for each n, G_n is a collection of regions covering S;* (2) *for each n, G_{n+1} is a subcollection of G_n;* (3) *if R is any region whatsoever, X is a point of R, and Y is a point of R either identical with X or not, then there exists a natural number m such that if g is any region belonging to the collection G_m and containing X then $\overline{g}$ is a subset of* $(R-Y)+X$; (4) *if $M_1, M_2, M_3, \cdots$ is a sequence of closed point sets such that, for each n, M_n contains M_{n+1} and, for each n, there exists a region g_n of the collection G_n such that M_n is a subset of $\overline{g}_n$, then there is at least one point common to all the point sets of the sequence $M_1, M_2, M_3, \cdots$.*

AXIOM 2. *If P is a point of a region R there exists a non-degenerate connected domain containing P and lying wholly in R.*

[‡] If M denotes a point set, $\overline{M}$ denotes the point set consisting of M and all the limit points of M.

[§] See Theorem 40 of Chapter II of *Foundations* and G. T. Whyburn, *On the cyclic connectivity theorem*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 429–433.

[‖] Theorem 1 of Chapter II of *Foundations*: *If A and B are distinct points of a connected domain D, there exists a simple continuous arc from A to B that lies wholly in D.*

an arc from A to B lying in $S-X$, and in this arc there exists a segment $\widehat{PE}$ lying in R such that $AB\cdot PE=P$ and E belongs to the boundary of R.† Therefore, if for no X of α is P identical with A, there exist a subsequence $X_1, X_2, X_3, \cdots$ of α and a sequence $P_1E_1, P_2E_2, P_3E_3, \cdots$ of arcs such that for each n (1) $AB\cdot P_nE_n=P_n$, (2) E_n belongs to the boundary of R, (3) $\widehat{P_nE_n}$ is a subset of R, and (4) X_{n+1} separates $\sum_{i=1}^{n}\widehat{P_iE_i}$ from A in R. But (1), (2), and (3) lead to a contradiction of (4), for since there exists in R a domain D containing A whose boundary is a subset of the sum of a finite collection Δ of continua lying in $R-D$, there exists a continuum H of Δ lying in $R-A$ such that H does not contain more than a finite number of the points $X_1, X_2, X_3, \cdots$ but for infinitely many values of n, H contains a point of P_nE_n. Therefore, either there exists an arc PE which is a subset of an arc from A to B such that $PE\cdot AB=P=A$, or if R is a region containing A but not B, there exists a point O of $AB\cdot R$ such that no point of the interval AO of AB separates A from O in R.

Suppose the latter is true. Let $R_1, R_2, R_3, \cdots$ be a monotonic sequence of regions closing down on A and $O_1, O_2, O_3, \cdots$ be a sequence of points of AB such that, for each n, (1) the interval AO_n of AB is a subset of R_n, (2) no point X of AO_n separates A from O_n in R_n. For each n and each point X of AO_n, there exists an arc T_{xn} lying in R_n and having only its end points in common with AB such that the segment S_{xn} of AB between these end points contains X and is a subset of R_n. For each n, there exists a finite collection of the segments S_{xn} covering the arc $O_{n+1}O_{n+2}$ of AB. But each $S_{xn}+T_{xn}$ is a simple closed curve lying in R_n. Hence there exists a sequence $J_1, J_2, J_3, \cdots$ of simple closed curves such that (1) for each n, J_n and J_{n+1} have at least two points in common, and (2) if R is a region containing A there exists a number δ such that if $n>\delta$, J_n is a subset of R. By Theorem 35 of Chapter II of *Foundations* there exists in $\sum J_n$ a simple closed curve J_A containing A and a segment of AB.

If, on the other hand, there exists an arc PE which is a subset of an arc PB from A to B such that $PE\cdot AB=P=A$, then there exists in PB an arc PX_1 such that P and X_1 are the only points PX_1 has in common with AB. But PX_1+AX_1(of AB) forms a simple closed curve containing A and some segment of AB.

So in either case there exists a simple closed curve J_A containing A and a segment $\widehat{OF}$ of AB. Likewise, there exists a simple closed curve J_B containing B and a segment $\widehat{YE}$ of AB. Now for each point X of the interval FY of AB, there exists an arc T_x such that T_x has only its end points in common with AB, and the segment S_x of AB lying between these end points contains X.

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† If AB is an arc with end points A and B, $\widehat{AB}$ is used to denote $AB-(A+B)$.

Some finite collection of these segments S_x covers the arc FY. Since every S_x+T_x is a simple closed curve, there exists a sequence of simple closed curves $J_1=J_A, J_2, \cdots, J_k=J_B$ such that for each n, $n<k$, J_n contains at least two points of J_{n+1}. By Theorem 35 of Chapter II of *Foundations* there exists in the sum of these simple closed curves a simple closed curve J containing A and B.†

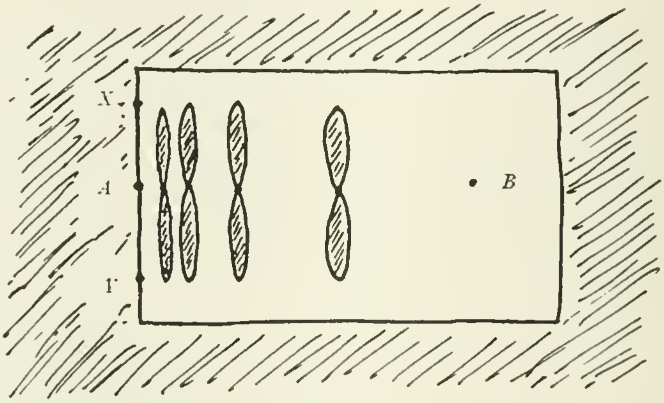

FIGURE 1

The example indicated in Figure 1 is a subset of a plane with the shaded portions removed. Except for those points of the arc XAY different from A, the boundary points of these shaded portions are not removed. It will be easily seen that the result is a connected, connected im kleinen inner limiting subset of the plane and hence satisfies Axioms 0, 1, and 2. Also Axiom 5_1^* is satisfied at every point except A. As a matter of fact, the space is locally compact at every point except A. It is also true in this space that if P is a point and R is a region containing A, there exists in R an arc (or a simple closed curve) separating A from P. The space contains no cut points. Yet there exists no simple closed curve containing A and B.

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† This argument may be slightly modified to establish a proposition somewhat more general than Theorem A.

THEOREM A′. *If* (1) *S is a space satisfying R. L. Moore's Axioms* 0, 1, *and* 2, (2) *neither of the distinct points A and B of S is a cut point nor a local end point of S, and* (3) *no point separates A from B in S, then A and B lie on a simple closed curve in S.* (See G. T. Whyburn, *The cyclic and higher connectivity of locally connected spaces*, American Journal of Mathematics, vol. 53 (1931), pp. 427–442.)

The modification may be outlined as follows: In the first paragraph of the proof, if A is not a local end point, $\prod V_X$ contains a point E' distinct from A which belongs to the component of R that contains A. Let AE' denote an arc from A to E' in R and let $E'B$ denote an arc from E' to B in $S-A$. Then $AE'+E'B$ contains an arc PE which is a subset of an arc from A to B such that $PE\cdot AB=P=A$. This takes us up to the last sentence of the first paragraph of the proof, which remains unchanged from there on.

Suppose now that, in addition to Axioms 0–2, S satisfies R. L. Moore's Axiom 4† and Axiom 5_1^*.

THEOREM B. *The space S is either acyclic or contains no cut point.*

Proof. Suppose that S contains a simple closed curve J. Then it is evident from Axiom 4 that S is connected. Suppose that some point X separates S. The point X does not belong to J, for it is clear from Axiom 4 that no point of a simple closed curve separates S. Let AX denote an arc with end points A and X such that $AX \cdot J = A$. Let M denote the set of cut points of S that belong to AX and let B denote the first point of $\overline{M}$ in the order from A to X. The point B is a cut point of S. For if it is not, then B is a sequential limit point of a sequence $B_1, B_2, B_3, \cdots$ of M and at the same time lies on a simple closed curve C containing a segment of AX which contains B; hence, C contains points of the sequence $B_1, B_2, B_3, \cdots$ which contradicts Axiom 4. But since the segment AB of AX contains no point of M, no point separates A from B in X and by Theorem A there exists a simple closed curve containing A and B which again contradicts Axiom 4.

It is not the purpose of this paper to treat the acyclic case;‡ so we shall at this point assume R. L. Moore's Axiom 3.§

1. CONSEQUENCES OF AXIOMS $0-4$ AND 5_1^*.

THEOREM 1. *If A and B are distinct points, there exists a simple closed curve containing A and B.*

THEOREM 2. *If A and B are distinct points, there exists a simple closed curve separating A from B.*

Proof. There exists a simple closed curve J such that J is the sum of two arcs AXB and AYB. Since $S-J$ is the sum of two connected domains I and E each having J for its boundary, there exist two arcs $X_1O_1Y_1$ and $X_2O_2Y_2$ such that (1) the points X_1 and X_2 lie on the segment AXB and the points Y_1 and Y_2 lie on the segment AYB, and (2) the segment $X_1O_1Y_1$ is a subset of I and the segment $X_2O_2Y_2$ is a subset of E. The simple closed curve formed by the sum of the four arcs $X_1O_1Y_1$, $X_2O_2Y_2$, X_1X_2(of AXB), and Y_1Y_2(of AYB) separates A from B in S.

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† If J is a simple closed curve, $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.

‡ Such a space if connected would be a *Regular* (*Menger*) *Curve*, and these curves have been studied considerably already. See *Foundations* for the definition and references on *Regular* (*Menger*) *Curves*.

§ If O is a point, $S-O$ is connected.

THEOREM 3. *If P is a point not belonging to the closed and compact point set H, there exists a simple closed curve separating P from H.*

Proof. For each point X of H there exists a simple closed curve separating P from X. Since H is closed and compact, there exists a finite collection of simple closed curves $J_1, J_2, \cdots, J_n$ such that if X is any point of H some one of them separates P from X. For each integer i, $i \leqq n$, let T_i denote an arc from J_1 to J_i lying in $S-P$ and let M denote the compact continuum $J_1+J_2+\cdots+J_n+T_1+T_2+\cdots+T_n$. The continuum M separates P from $H-H\cdot M$. Now for each point X of M there is a simple closed curve separating X from P. Let $C_1, C_2, \cdots, C_j$ denote a finite collection of these whose interiors with respect to P as the point at infinity† cover M. By Theorem 13 of Chapter III of *Foundations*, $C_1+C_2+\cdots+C_j$ contains a simple closed curve separating P from M and hence, separating P from H.

With the help of Theorem 3, the arguments of R. L. Moore with slight modifications prove that a number of the theorems of Chapter IV of *Foundations* hold true in this space. Of these the following five theorems will be used.

THEOREM 4. *If the points A and B belong to different components of the closed and compact point set M, there exists a simple closed curve separating A from B and containing no point of M* (Theorem 10 of Chapter IV of *Foundations*).

THEOREM 5. *If α and β are two connected point sets and neither of the two mutually exclusive closed and compact point sets H and K separates α from β, then $H+K$ does not separate α from β* (Theorem 16).

THEOREM 6. *If the common part of the closed and compact point sets H and K is a continuum and neither H nor K separates the point A from the point B, then $H+K$ does not separate A from B* (Theorem 18).

THEOREM 7. *No arc separates S* (Theorem 19).

THEOREM 8. *If the points A and B are separated from each other by the closed and compact point set M then they are separated from each other by a continuum which is a subset of M and which contains no proper subset that separates A from B* (Theorem 24).

THEOREM 9. *If the compact continuum K does not separate the point A from the point B and G is a finite collection of compact continua such that* (1) *the common part of any two elements of G is a subset of K, and* (2) *if H is any element of G, H does not separate A from B and $H\cdot K$ is either connected or vacuous, then $K+G^*$ does not separate A from B.*

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† For the definition of certain terms and phrases used without definition in this paper the reader is referred to *Foundations*.

Theorem 9 may be established by a finite number of applications of Theorems 5 and 6.

THEOREM 10. *If AXB is an arc and J is a simple closed curve separating A from B, then $J+AXB$ contains a simple closed curve J_1 separating A from B such that $J_1 \cdot AXB$ is connected.*

Proof. Let A_1 and B_1 denote the first and last points respectively that the arc AXB has in common with J. Let AYB denote an arc from A to B not containing any point of the interval A_1B_1 of AXB. Let G denote the collection of all simple closed curves C in $J+AXB$ such that $C \cdot AXB$ is connected and $C \cdot AYB$ is not vacuous. The collection G is finite. Let K denote $A_1B_1+(J-J \cdot G^*)$.[†] The point set K is a continuum containing no point of AYB; hence, K does not separate A from B. Since the common part of any two elements of G is a subset of K and the common part of K with an element of G is connected, if no simple closed curve of the collection G separated A from B, then by Theorem 9, $K+G^*$ would not separate A from B. But $K+G^*$ contains J and separates A from B. Therefore some element J_1 of G separates A from B.

DEFINITIONS. Suppose that P is a point of a domain Q, and Δ is a collection of continua whose sum separates P from the boundary of Q such that each continuum of Δ lies in a component of $Q-P$ whose boundary contains P but no component of $Q-P$ contains more than one element of Δ. Then (1) Δ is said to be *minimal*[‡] with respect to Q and P, (2) Δ is said to *minimally separate* P from the boundary of Q, and (3) if D is a domain containing P whose boundary is a subset of Δ^*, Δ is said to *surround D minimally* with respect to Q and P.

THEOREM 11. *If Δ is a collection of continua which surrounds a connected domain D minimally with respect to a domain Q and a point P, then* (a) *each component of $Q-P$ whose boundary contains P, contains one and only one element of Δ, and* (b) *no component of $D-P$ has boundary points in more than one element of Δ.*

Proof. Consider (a). By definition, if C is a component of $Q-P$ whose boundary contains P, then C contains not more than one element of Δ. But since $S-P$ is connected, the boundary of C also contains a point of the boundary of Q; hence C must contain at least one element of Δ. Therefore C contains one and only one element of Δ.

[†] If G denotes a collection of point sets, G^* will be used to denote the sum of the elements of G.

[‡] The term *minimal* is used here because if Δ were finite no collection of fewer elements would separate P from the boundary of Q.

Next consider (b). Suppose that there exists a component I of $D-P$ which has boundary points in more than one element of Δ. Let C denote the component of $Q-P$ which contains I. Then C contains more than one element of Δ. But since D is connected, the boundary of C contains P. This is a contradiction of (a).

THEOREM 12. *If P is a point of a domain Q, not S, there exist in Q a finite collection Δ of continua and a connected domain D containing P such that Δ surrounds D minimally with respect to Q and P.*

Proof. Let D_1 denote a domain lying in Q and containing P whose boundary is a subset of the sum of the elements of a finite collection Δ_1 of continua lying in $Q-D_1$. Let N denote the sum of all the continua of Δ_1 which lie in components of $Q-P$ whose boundaries contain P. If H and K are components of N lying in the same component C of $Q-P$, let T denote an arc in C from a point of H to a point of K. Let Δ denote the collection of all components of the point set obtained by adding N to the sum of the arcs T. The collection Δ minimally separates P from the boundary of Q. Let D denote the component of $Q-\Delta^*$ which contains P. Then the boundary of D is a subset of Δ^* and hence Δ surrounds D minimally with respect to Q and P.

DEFINITION. If a domain is connected and contains one of the complementary domains of each simple closed curve lying in it, then it is said to be *simply connected*.

DEFINITION. If D is a connected domain and D contains one of the complementary domains of each simple closed curve lying in $\overline{D}$, then $\overline{D}$ is said to be *simply connected* and D is said to be *strongly simply connected*.

THEOREM 13. *Space is simply connected.*

THEOREM 14. *If D is a connected domain and there exist a collection Δ of continua, a domain Q, and a point P of D, such that Δ surrounds D minimally with respect to Q and P, then D is simply connected.*

Proof. Suppose, on the contrary, that there exists in D a simple closed curve J such that neither of its complementary domains, I and E, is a subset of D. Since J lies in D, both I and E contain points of D and, therefore, points of the boundary β of D. Let T denote an arc lying in $S-P$ irreducible from $\beta\cdot I$ to $\beta\cdot E$. The segment T intersects J and, hence, is a subset of D. Therefore T lies in $Q-P$ and contains points of two different continua of Δ. This contradicts Theorem 11.

THEOREM 15. *If P is a point of a domain Q, there exists in Q a simply connected domain D containing P.*

Theorem 15 is a consequence of Theorems 12, 13, and 14.

THEOREM 16. *If D is a connected domain, every component of $S-\overline{D}$ is strongly simply connected.*

Proof. Let C denote a component of $S-\overline{D}$. Let J denote a simple closed curve lying in $\overline{C}$, and let I denote the interior of J with respect to a point ω of D. Suppose that X is a point of $\overline{D}$ and belongs to I. Then $X+I$ is a connected point set containing ω and a point of I but no point of J. This is a contradiction.

THEOREM 17. *If D is a complementary domain of a locally compact continuous curve M and P is a point of the boundary of D, then $K=D+P$ is a connected, connected im kleinen inner limiting set.*

Proof. It is evident that K is a connected inner limiting set. It is furthermore evident that K is connected im kleinen at every one of its points with the possible exception of P. Suppose that K is not connected im kleinen at P. Then there exists a simply connected domain Q containing P such that (1) the component C of $Q\cdot K$ that contains P is not open with respect to K at the point P, and (2) $Q\cdot M$ is compact. Hence there exists a sequence α of points of K converging to P such that each point of α belongs to some component of $Q\cdot K$ but not to C and no two points of α belong to the same component of $Q\cdot K$. There exists a point O of K not belonging to Q, for otherwise $Q\cdot K=K$ would be connected and open with respect to K at P. For each point X of α let XO denote an arc from X to O lying in D, and let Y denote the first point in the order from X to O that XO has in common with the boundary of Q.

Now let Q_1 denote a domain lying in Q and containing P such that β_1, the boundary of Q_1, is a subset of the sum of a finite collection Δ of continua lying in $Q-Q_1$. For infinitely many points X of the sequence α the interval XY of the arc XO contains a point of some one of the continua of the finite collection Δ. Hence there exists a continuum H of Δ such that for infinitely many points X of α the arc XY contains a point of H. Now let D_H and D_P denote connected domains lying in Q and containing H and P respectively such that $\overline{D}_H\cdot\overline{D}_P=0$.

For each point X of α such that X lies in D_P and the arc XY contains some point of H, let TX denote an arc lying in D_P such that T belongs to M but the segment TX is a subset of D, and let Z denote a point of $XY\cdot H$. Now since D_H lies in Q and contains points of more that one component of $Q\cdot K$, D_H contains some point of M. For each Z let ZW denote an arc lying in D_H such that the segment ZW is a subset of Q and W is a point of M. Thus for some infinite subsequence $x_1, x_2, x_3, \cdots$ of α there correspond sequences of points $T_1, T_2, T_3, \cdots$ and $W_1, W_2, W_3, \cdots$ such that (1) the points $T_1, T_2, T_3, \cdots$ belong to $D_P \cdot M$, (2) the points $W_1, W_2, W_3, \cdots$ belong to $D_H \cdot M$, (3) for each n, $T_nX_n+X_nZ_n+X_nW_n$ contains an arc T_nW_n lying in Q, and (4) if $m \neq n$, $\widehat{T_mW_m}$ and $\widehat{T_nW_n}$ are mutually exclusive and lie in different components of $Q \cdot K$. Since M is a continuous curve, $Q \cdot M$ is compact and $\overline{D}_H \cdot \overline{D}_P = 0$, there exist two mutually exclusive connected subsets d_P and d_H of $Q \cdot M$ which are open with respect to M such that for infinitely many values of n and in particular for the three different values a, b, and c of n, T_n and W_n lie in d_P and d_H respectively. Let T_aT_b and T_aT_c denote arcs lying in d_P and W_aW_b and W_aW_c denote arcs lying in d_H. There exist three arcs AO_aB, AO_bB, and AO_cB lying in $T_aT_b+T_aW_a+W_aW_b$, $T_aT_b+T_bW_b+W_aW_b$, and $T_aT_c+T_cW_c+W_aW_c$ respectively and having only their end points A and B in common. By Theorem 5 of Chapter III of *Foundations*, the sum of one pair of these three arcs, say AO_aB and AO_cB, forms a simple closed curve J lying in Q and whose interior I with respect to a point ω of the boundary of Q contains the other segment, $\widehat{AO_bB}$. Hence I contains the component C of $Q \cdot K$ which contains $\widehat{T_bW_b}$, for C contains no point of J. However, C has limit points in the boundary of Q. But since Q is simply connected, $\bar{I}$ is a subset of Q. This is a contradiction, and K is connected im kleinen at P.

THEOREM 18. *If D is a complementary domain of a locally compact continuous curve, and P is a point of the boundary of D, then P is accessible from D.*

Theorem 18 follows immediately from Theorems 1 and 10 of *Foundations* and Theorem 17.

A number of the intuitive propositions of the plane concerning abutting and crossing arcs hold true. Although some of these will be used in arguments to follow, they will not be stated and the reader is referred to Chapter IV, Theorems 28–32, of *Foundations* for their precise statement and proof. Some of these proofs must be modified, however, to be valid for the set of axioms used here.

THEOREM 19. *An arc is accessible from both sides at any interior point.*

THEOREM 20. *A ray separates space into at most two connected domains.*

Proof. Suppose, on the contrary, that there exists a ray $\overrightarrow{PB}$ such that $S-\overrightarrow{PB}$ contains three components. There exist three arcs A_1B_1, A_2B_2, and A_3B_3 no two of which contain points of the same component of $S-\overrightarrow{PB}$ such that (1) for each i, $(i=1, 2, 3)$, $A_iB_i \cdot \overrightarrow{PB} = B_i$, and (2) B_1 belongs to the interval PB_2, and B_2 to the interval PB_3, of $\overrightarrow{PB}$. By Theorems 7 and 9, $M=A_1B_1+A_2B_2+A_3B_3+PB_3$ does not separate space. By Theorem 18 the points A_1, A_2, and A_3 are accessible from $S-M$. Thus for each i, $i=1, 2, 3$, there exists an arc A_iB_i' lying in $(S-M)+A_i$ and having only the point B_i' in common with $\overrightarrow{PB}$. For each i, $i=1, 2, 3$, let J_i denote the simple closed curve composed of A_iB_i, A_iB_i' and the interval B_iB_i' of $\overrightarrow{PB}$, and let I_i denote that complementary domain of J_i whose common part (if any) with $\overrightarrow{PB}$ is compact. Either (1) I_1 contains A_2, or (2) I_3 contains A_2, or (3) I_2 contains A_1, or (4) I_2 contains A_3. Any one of these four possibilities leads to a contradiction. For instance, suppose that I_1 contains A_2. Since $I_1\cdot\overrightarrow{PB}$ is a subset of PB_1, it is clear, by Theorems 9 and 18, that I_1+A_1 contains an arc from A_1 to A_2 not intersecting $\overrightarrow{PB}$.

Theorem 21. *If the arc AB is a subset of the connected domain D and $D-AB$ is connected, then D contains a simple closed curve separating A from B.*

Proof. Since the arc AB is accessible from both sides at an interior point O, there exist two arcs EO and FO abutting on AB from different sides such that neither EO nor FO has any point except O in common with the arc AB. The arcs EO and FO contain points E_1 and F_1, respectively, such that the intervals E_1O and F_1O of EO and FO respectively lie in D. In $D-AB$ there exists an arc E_1F_1, and in $E_1O+E_1F_1+F_1O$ there exists a simple closed curve J which lies in D and separates A from B.

Theorem 22. *If P is a point of a connected domain Q, then there exists a region R containing P such that if X is a point of $R-P$, X lies in a simple domain which together with its boundary is a subset of Q.*

Proof. If Q is S, the theorem is evident from Theorem 2. Suppose that the boundary of Q is not vacuous and that the theorem is false.

Then there exists a sequence α of points X of $Q-P$ converging to P such that X does not lie in a simple domain which together with its boundary is a subset of Q. Let ρ denote a monotonic sequence of simply connected domains D closing down on P. Let D_1 denote an element of ρ lying in Q. It is clear from Axioms 3 and 5_1^* that Q_1-P contains only a finite number of components. One of these components C_1 contains two points X_1 and X_2 of α. Let T_1 denote an arc from X_1 to X_2 in C_1. Let D_2 denote the first element of ρ which follows D_1 and contains no point of T_1. One of the components of D_2-P contains two points X_3 and X_4 of α and an arc T_2 from X_3 to X_4. Let D_3 denote the first element of ρ not containing a point of T_1+T_2. Then D_3-P contains an arc T_3 whose end points are points of α. This process may be continued. Thus there exists a sequence of mutually exclusive arcs $T_1, T_2, T_3, \cdots$ converging to P such that for each n, T_n is a subset of D_n-P and the end points of T_n are points of α. By the preceding theorem, for each n, T_n separates D_1.

For each n, one of the components of $D_1 - T_n$ does not contain P and in this component there exists a segment W_n having one end point in T_n and containing no point of $T_1 + T_2 + \cdots + T_{n-1}$. By Theorems 7 and 9, $P + T_1 + T_2 + \cdots + T_n$ does not separate space. Hence there exists a segment W_n' lying in $D_1 - (P + T_1 + T_2 + \cdots + T_n)$ having one end point in W_n and the other end point in the boundary of D_1. For each n let L_n denote the component of $D_1 - (P + T_1 + T_2 + \cdots + T_n)$ which contains $W_n + W_n'$. Then for each n, T_n separates L_n from P in D_1, and $\overline{L}_n$ contains points of both T_n and the boundary of D_1. By Axiom 5_1^* there exists in D_1 a domain U which contains P and whose boundary is a subset of the sum of a finite number of continua lying in $D_1 - U$. There is a continuum H of this set which, for infinitely many values j of n, contains a point of L_j while U contains T_j. But if k is the smallest value of j, $P + H + \sum L_j$ is a connected point set in $D_1 - T_k$ containing P and L_k. This is a contradiction.

Definitions. If P is a point and there exists a sequence of simple domains closing down on P, then P is said to be a *simple point*. A non-simple point is said to be an *edge point*.

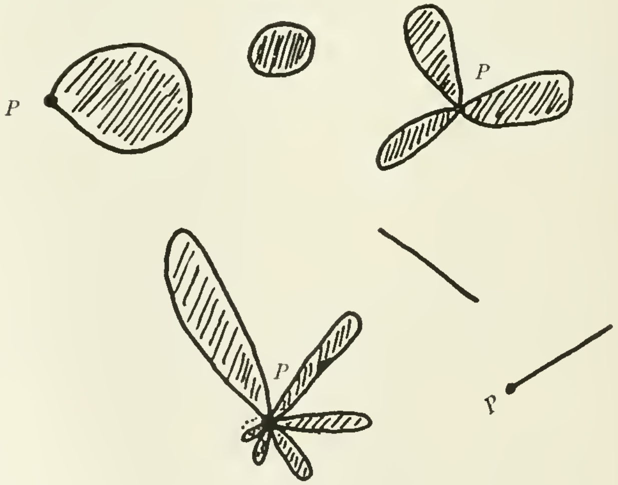

Figure 2

If from the plane of Figure 2 the shaded domains and their boundaries except for the points P are removed, the result is a space which satisfies Axioms 0–4 and 5_1^*. The points P are *edge points* while all other points are *simple points*.

Theorem 23. *The set of simple points is everywhere dense.*

Proof. Suppose that R is a region. With the help of repeated applications of Theorem 22 it may be shown that there exists a sequence $D_1, D_2, D_3, \cdots$ of simple domains lying in R such that, for each n, (1) D_n contains $\overline{D}_{n+1}$, and (2) $\overline{D}_n$ lies in some region of the collection G_n of Axiom 1. By (4) of Axiom 1, there is a point P common to $\overline{D}_1, \overline{D}_2, \overline{D}_3, \cdots$ and hence, to $D_1, D_2, D_3, \cdots$; and by (3) of Axiom 1, P is the only common point. The point P is simple and lies in R.

Theorem 24. *No completely separable point set contains uncountably many edge points.*

Proof. Let H denote the collection of all edge points belonging to a completely separable point set M. If X is a point of H, there exists no sequence of simple domains closing down on X. Hence for each point X of H, there exists an integer n_x such that no region of G_{n_x} of Axiom 1 contains a simple domain containing X. If H is uncountable, there exist an integer k and an uncountable subset K of H, such that if X is a point of K, $n_x = k$. Since M is completely separable, K has a point P of condensation. Let Q denote a connected domain containing P and lying in a region of G_k of Axiom 1. By Theorem 22 there exists in Q a region R containing P such that every point X of $R-P$ is contained in a simple domain lying in Q. This is clearly a contradiction.

Theorem 25. *If M is a countable set of simple points, $S-M$ is arc-wise connected.*

This theorem may be proved without using Axioms 4 and 5_1^* by a modification of R. L. Moore's proof[†] of Theorem 1 of Chapter II of *Foundations*.[‡] The kernel of this proof is the construction of a sequence $C_1, C_2, C_3, \cdots$ of simple chains of connected domains from a point A to a point B such that for each n, C_n^* contains $\overline{C}_{n+1}^*$ and the common part of the point sets $C_1^*, C_2^*, C_3^*, \cdots$ is an arc from A to B. Now for each n, C_n^* is a connected domain, and it is easy to see that no finite number of simple points disconnects a connected domain. Hence if (1) A and B are two distinct points of $S-M$, (2) $M = P_1 + P_2 + \cdots + P_n + \cdots$, and (3) for each n the chain C_n is constructed so that C_n^* contains no point of $P_1 + P_2 + \cdots + P_n$, then the arc from A to B of Moore's construction will not contain any point of M.

Theorem 26. *If M is a countable set of simple points, $S-M$ is cyclicly connected.*

Proof. If A and B are any two points of $S-M$, then by Theorem 25 there exists an arc AXB in $S-M$. With the help of Theorems 5, 7 and 17, and

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[†] Cf. J. R. Kline, *Concerning the complement of a countable infinity of point sets of a certain type*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 290–292.

[‡] See footnote p. 54.

Theorem 9 of Chapter II of *Foundations*, one may show that $S-\widehat{AXB}$ is a space satisfying Axioms 0, 1, 2, and 3. Hence by Theorem 25 there exists an arc AYB lying in $S-(M+\widehat{AXB})$. The sum of the two arcs, AXB and AYB, is a simple closed curve lying in $S-M$ and containing $A+B$.

THEOREM 27. *No compact set of edge points separates two simple points from each other.*

Proof. Suppose that A and B are two simple points and that M is a compact set of edge points. Suppose that n is any fixed integer. Then there exists from A to B a simple chain of simple domains such that each element of the chain is a subset of some region of G_n of Axiom 1. For suppose the contrary. Let H_A denote the set of all points X such that there exists from A to X a chain of simple domains whose elements are each a subset of some region of G_n. Since A is a simple point, H_A is a connected domain. This domain does not contain B, and its boundary β is a subset of M. Hence β is a closed and compact point set which separates A from B. By Theorem 8 and Axiom 3, β contains a non-degenerate continuum separating A from B. Hence M is uncountable, which contradicts Theorem 24.

Now with the help of Theorem 9 of Chapter II of *Foundations* and Theorem 5 one may show that a simple domain is a space satisfying Axioms 0–4 and 5_1^*, and that the complementary domain of a point is also such a space. Let $M=P_1+P_2+P_3+\cdots$. Let C_1 denote a simple chain of simple domains from A to B such that (1) each element of C_1 is a subset of some region of G_1, and (2) C_1^* does not contain P_1. Now the boundary of the domain C_1^* is a closed and compact point set not separating A from B; so there exists a simple closed curve separating it from $A+B$ (Theorem 3 in modification). Hence there exists in C_1^* a simple domain D_1 containing $A+B$. Let C_2 denote a simple chain of simple domains from A to B such that if c is an element of D_2, (1) $\bar{c}$ is a subset of some element of C_1, (2) c is a subset of some region of G_2, and (3) c does not contain P_2. This process may be continued, and by Theorem 80 of Chapter I of *Foundations*, $C_1^*\cdot C_2^*\cdot C_3^*\cdot\cdots$ is a continuum. Hence M does not separate A from B.

As a matter of fact the above process may be carried out along the lines of the proof of Theorem 1 of Chapter II of *Foundations*, so as to show that $S-M$ is arc-wise, and even cyclicly, connected.

Nevertheless, a countable but non-compact set of edge points may separate two simple points from each other as is shown by the following example. Let space consist of all points of the number plane whose ordinates are not zero together with those whose ordinates are zero but whose abscissas are rational. If (X, Y) is a point whose ordinate Y is not zero, then for each integer such that $1/n < |Y|$, let $V_n(X, Y)$ denote the point set consisting of all points which lie at a distance less than $1/n$ from (X, Y). If (X, Y) is a point whose ordinate Y is zero and whose abscissa X is rational then for each integer n, let $V_n(X, Y)$ denote the point set consisting of (X, Y) together with all points which are interior to the isosceles triangles bounded by the lines whose equations are $y=1/n$, $y=-1/n$, $y=n(x-X)$, and $y=-n(x-X)$. These point sets V_n shall be called regions, and for each integer n, G_n of Axiom 1 shall denote the collection of all of the point sets V_{n+q}, $q=0, 1, 2, \cdots$. It can be shown that Axioms 0–4 and 5_1^* hold true in this space. Furthermore, the space is completely separable and locally compact at all but a countable nowhere-dense set of points. Let M denote the set of all points whose abscissas are rational and whose ordinates are zero. Each point of M is an edge point. If A and B are points with positive and negative ordinates respectively, M separates A from B, and the point set H_A of the above argument consists of all points whose ordinates are positive.

THEOREM 28. *No compact set of edge points separates space.*

Theorem 28 is a consequence of Theorems 23 and 27.

DEFINITIONS. If D_1 and D_2 are domains such that D_1 contains $\overline{D}_2$, and $\widehat{T}$ is a segment lying in $D_1-\overline{D}_2$ having one end point on the boundary of D_1 and the other on the boundary of D_2, then $\widehat{T}$ is said to *cross* $D_1-\overline{D}_2$. Furthermore, if C is the component of $D_1-\overline{D}_2$ which contains $\widehat{T}$, then $\widehat{T}$ is said to *cross* C. Also the arc T is said to *cross* $D_1-\overline{D}_2$ and to *cross* C.

DEFINITION. If D_1 and D_2 are domains such that D_1 contains $\overline{D}_2$, C is a component of $D_1-\overline{D}_2$, and $\widehat{T}_1$, $\widehat{T}_2$, and $\widehat{T}_3$ are segments crossing C, then if $\widehat{T}_2$ separates $\widehat{T}_1$ from $\widehat{T}_3$ in C, $\widehat{T}_2$ is said to be *between* $\widehat{T}_1$ and $\widehat{T}_3$ in C, and T_2 is said to be *between* T_1 and T_3 in C.

THEOREM 29. *Suppose that D_1 and D_2 are simply connected domains such that D_1 contains $\overline{D}_2$ but no simple domain lying in D_1 contains D_2, C is a component of $D_1-\overline{D}_2$, and $\widehat{T}_1$, $\widehat{T}_2$, and $\widehat{T}_3$ are three mutually exclusive segments crossing C. Then one and only one of these segments is between the other two in C.*

Proof. That not more than one of these segments lies between the other two in C is an immediate consequence of a well known theorem.

Suppose that no one of them is between the other two in C. Then either (1) $C-(\widehat{T}_1+\widehat{T}_2+\widehat{T}_3)$ contains a component W which has limit points in each of the three segments, or (2) $C-(\widehat{T}_1+\widehat{T}_2+\widehat{T}_3)$ contains three mutually exclusive components W_{12}, W_{13}, and W_{23} having limit points in the segments indicated by their subscripts.

Case I. Suppose (1) holds. Then there exists in W three mutually exclusive segments $\widehat{AA_1}$, $\widehat{AA_2}$, and $\widehat{AA_3}$ having a common end point A in W such that A_1, A_2, and A_3 belong to $\widehat{T}_1$, $\widehat{T}_2$, and $\widehat{T}_3$ respectively. For each i, $i=1, 2, 3$, let O_i and P_i denote the end points of T_i which belong to the boundaries of D_1 and D_2 respectively. Since D_2 is connected, there exist in $D_1-(\widehat{T}_1+\widehat{T}_2+\widehat{T}_3+AA_1+AA_2+AA_3)$ three mutually exclusive segments $\widehat{BB_1}$, $\widehat{BB_2}$, and $\widehat{BB_3}$ having a common end point B in D_2 and having their other end points B_1, B_2, and B_3 in the intervals A_1P_1 of T_1, A_2P_2 of T_2, and A_3P_3 of T_3 respectively. If AB_nB denotes $AA_n+A_nB(\text{of } A_nP_n)+BB_n$, $n=1, 2, 3$, then the segments AB_1B, AB_2B, and AB_3B are mutually exclusive. By Theorem 5 of Chapter III of *Foundations*, the sum of two of these arcs, say AB_1B and AB_3B, forms a simple closed curve J whose interior I with respect to O_1 contains the segment from A to B of the other one. Since D_1 is simply connected, I is a subset of D_1. But T_2 contains B_2 and no point of J and is, therefore, a subset of I. Hence I contains O_2. This is a contradiction.

Case II. Suppose (2) holds. There exist segments $\widehat{A_1B_3}$, $\widehat{B_2A_3}$, and $\widehat{B_1A_2}$ lying in W_{13}, W_{23}, and W_{12} respectively and having end points on the arcs T_1, T_2, and T_3 as indicated by the subscripts in the notation. The point set $A_1B_3+A_1B_1(\text{of } T_1)+B_1A_2(\text{of } T_2)+B_2A_3+A_3B_3(\text{of } T_3)$ is a simple closed curve J lying in D_1-D_2. Since D_1 is simply connected, one of the complementary domains I of J is a subset of D_1. The domain I either contains D_2 or is a subset of $D_1-\overline{D}_2$. By hypothesis D_1 contains no simple domain containing D_2. Hence I is a subset of $C-(\widehat{T}_1+\widehat{T}_2+\widehat{T}_3)$ and has limit points in each of the three segments $\widehat{T}_1$, $\widehat{T}_2$, and $\widehat{T}_3$, which is Case I again.

The reader can see that any collection of mutually exclusive segments crossing C would, in accordance with this notion of *between*, have a *linear order* in C.

THEOREM 30. *Suppose that D_1, D_2, and D_3 are simply connected domains containing the point P such that* (1) *D_3 contains $\overline{D}_2$,* (2) *D_3 is surrounded minimally with respect to D_1 and P by a finite collection Δ of continua lying in D_1-D_3, and* (3) *D_1 contains no simple domain containing P. Then if C is a component of $D_1-\overline{D}_2$, T is an arc crossing C, and J is a simple closed curve separating T from P, J contains two arcs T_a and T_b which cross C such that T is between them in C.*

Proof. Let R denote a connected domain lying in $D_1-D_1\cdot J$ and containing the end point A of T which is on the boundary of D_2. Now in $R+D_2$ there exists an arc AP from A to P, such that $J\cdot AP$ is a subset of an interval $A'P'$ of AP lying in D_2. By Theorem 10, $J+AP$ contains a simple closed curve J_1

separating A from P such that $J_1 \cdot AP$ is connected; but $J_1 \cdot AP$ is a subset of $A'P'$ and is, therefore, a subset of D_2. Since D_1 is simply connected but contains no simple domain containing P, J_1 contains a point O not belonging to D_1. The point set $J_1 - (J_1 \cdot AP + O)$ is the sum of two mutually exclusive segments $\overset{\frown}{OX}$ and $\overset{\frown}{OY}$, where X and Y belong to AP (and Y belongs to the interval XP of AP) and the arcs OX and OY abut on AP from different sides. Both OX and OY contain arcs which cross $D_1 - \bar{D}_2$. Let T_a denote the first interval of OX in the order from X to O that crosses $D_1 - \bar{D}_2$, and let T_b denote the first interval of OY in the order from Y to O which crosses $D_1 - \bar{D}_2$. Now C contains an element of Δ and if U denotes the component of $D_1 - P$ which contains C, U has no point in common with any other component of $D_1 - \bar{D}_2$ because Δ is a minimal collection with respect to D_1 and P. Hence T_a and T_b cross C. Let X' and Y' denote the end points of T_a and T_b respectively which are on the boundary of D_2. The intervals XX' of OX and YY' of OY lie in D_1.

Suppose that T is not between T_a and T_b in C. Then there exists an arc T_1 in C irreducible from T_a to T_b not intersecting T. Let W denote the last point that OX has in common with T_1, and let Z denote the last point that OY has in common with T_1. The intervals WX of OX and ZY of OY are subsets of $T_a + XX'$ and $T_b + YY'$ respectively, and hence, are subsets of D_1; furthermore, WX and ZY abut on AP from different sides. Now $J_2 = J_1 \cdot AP + WX + WZ$(of T_1)$+ZY$ is a simple closed curve lying in D_1 not containing P, or a point of T, or a point of the segment $\overset{\frown}{PY}$ of AP. Since D_1 is simply connected, one of the complementary domains I of J_2 is a subset of D_1. But since D_1 does not contain a simple domain containing P, both P and T must be subsets of the other complementary domain E of J_2. Now AX contains an arc which abuts on J_2 from the side opposite PY of AP, and since both A and $\overset{\frown}{PY}$ are in D, $\overset{\frown}{AX}$ must intersect J_2. Let A_1 and A_2 denote the first and last points respectively that $\overset{\frown}{AX}$ has in common with J_2. The segments $\overset{\frown}{AA_1}$ and $\overset{\frown}{A_2X}$ of AX lie in E and I respectively. The segment $\overset{\frown}{AX}$ does not intersect J_1; so $\overset{\frown}{AX} \cdot J_2$ is a subset of T_1. Hence $\overset{\frown}{AA_1}$ and $\overset{\frown}{A_2X}$ abut on T_1 from different sides. Let R' denote a connected domain lying in D_1 and containing A but no point of $T_1 + T_a + T_b$. Then in $R' + D_2$ there exists an arc FG irreducible from AA_1 to A_2X. The simple closed curve $J_3 = FG + GA_2$(of A_2X)$+A_1A_2$(of T_1)$+A_1F$(of AA_1) lies in D_1, contains no point of $T_a + T_b$, but crosses the arc T_1. Hence, T_a and T_b lie in different complementary domains of J_3. This is impossible, for since D_1 is simply connected, one complementary domain of J_3 is a subset of D_1.

THEOREM 31. *In order that space be metric, it is necessary and sufficient that space be completely separable.*

Proof. That a space satisfying Axioms 0 and 1 is metric if it is completely separable has been shown by R. L. Moore.† Hence, this space is metric if it is completely separable.

It will now be shown that if space is metric, it is completely separable. Let P denote an edge point, and let R denote a region containing P but containing no simple domain containing P. Since R is not S, by Theorems 12, 14, and 15, there exist in R three simply connected domains D_1, D_2, and D_3 such that D_3 contains $\overline{D}_2$, and D_3 is surrounded minimally with respect to D_1 and P by a collection Δ of continua lying in D_1-D_3. The domain D_1 contains no simple domain containing P. Let C denote a component of $D_1-\overline{D}_2$.

Suppose that there are two mutually exclusive arcs T_1 and T_{-1} crossing C. Let α denote a well-ordered sequence (whose first element is T_1) of all arcs T which cross C such that either (1) T is T_1, or (2) T_1 lies between T_{-1} and T in C. Let α' denote a subsequence of α such that (1) the first element of α' is T_1 and (2) if an element T of α' is not T_1, then T is the first element of α which neither intersects a preceding element of α' nor lies between two preceding elements of α'. Every element of α either intersects an element of α' or lies between T_1 and some element of α', and between any element of α' and the next following element in α' there is no element of α'.

The sequence α' is countable. For suppose that α' is uncountable. For each arc T of α' let d_T denote the distance from T to the next element of α'. There exist a number ϵ and an uncountable subsequence α'_ϵ of α' such that if T belongs to α'_ϵ, $d_T>\epsilon$. Let T_2 denote the first element of α' such that T is preceded in α' by infinitely many elements of α'_ϵ. Let L denote an arc from T_1 to T_2 lying in C. Infinitely many elements of α'_ϵ are between T_1 and T_2 in C and therefore intersect L. Consequently, there exists in C a connected domain D of diameter less than $\epsilon/2$ which contains points of two different elements T and T' of α'_ϵ, such that T precedes T' in α'. Then, the first element of α' which follows T in α' is either T' or lies between T and T' in C and intersects D, which is a contradiction. The same may be done for the collection of all arcs T such that T_{-1} lies between T_1 and T in C. Hence there exists a countable collection G of arcs crossing C, such that any arc which crosses C either intersects an arc of G or lies between two arcs of G. This was on the assumption that there exist two mutually exclusive arcs crossing C. It is evident that G exists if this is not the case, for there exists at least one arc which crosses C.

For each pair of arcs of G let L denote an arc in C which contains points of both of them, and let M_c denote the sum of all these arcs together with the sum of the elements of G. Then if T is any arc which crosses C, T intersects

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† See *Foundations*, p. 464.

M_c. Since each component C of $D_1-\overline{D}_2$ contains an element of Δ, there are only a finite number of them. Hence $M=\overline{\sum M_c}$ is a closed and completely separable subset of $\overline{D}_1-P$ which separates P from the boundary of D_1. Let Q denote the component of $\overline{D}_1-M$ which contains P. Then Q contains P and lies in R, and the boundary of Q is completely separable. Thus, if P is an edge point of a region R, there exists in R a domain containing P whose boundary is separable. This is also true if P is a simple point of a region R. Therefore, space is locally peripherally separable. By a theorem of the author's such a connected, connected im kleinen metric space is completely separable.†

THEOREM 32. *If space is completely separable and P is a point, there exists a sequence of simple domains $Q_1, Q_2, Q_3, \cdots$ bounded by simple closed curves $J_1, J_2, J_3, \cdots$ respectively, such that* (1) *for each n, Q_n contains $\overline{Q}_{n+1}$, and* (2) *if M is a closed and compact point set not containing P, there exists an integer n such that $\overline{Q}_n$ does not contain a point of M.*

Proof. Let G denote the collection of all simple domains D such that $\overline{D}$ is a subset of $S-P$. By Theorem 2, G covers $S-P$, and since space is completely separable, G contains a countable subcollection $D_1, D_2, D_3, \cdots$ which covers $S-P$. For each n let C_n denote the boundary of D_n. Let J_1 denote the boundary of a simple domain containing P. For each n, $n\geqq 2$, let J_n denote a simple closed curve separating P from the closed and compact point set $C_1+C_2+\cdots+C_{n-1}+J_1+J_2+\cdots+J_{n-1}$, and let Q_n denote the complementary domain of J_n which contains P. Then $Q_1, Q_2, Q_3, \cdots$ is the required sequence of simple domains. For if M is any closed and compact point set, there exists an integer k such that $D_1+D_2+\cdots+D_k$ covers M. Hence, if $n>k$, $\overline{Q}_n$ contains no point of M.

DEFINITION. Suppose that D is a connected domain. If there exist a collection Δ of open curves, a simply connected domain Q, and a point P of D, such that (1) Q contains no simple domain containing P, and (2) Δ surrounds D minimally with respect to Q and P, then D is said to be *pseudo-simple*, and in particular, D is said to be *pseudo-simple* with respect to Q and P.

THEOREM 33. *If space is completely separable and D_1 is a simply connected domain containing an edge point P but containing no simple domain containing P, then D_1 contains a domain D which is pseudo-simple with respect to D_1 and P.*

Proof. Let D_2 and D_3 denote simply connected domains containing P such that (1) D_3 contains $\overline{D}_2$, and (2) D_3 is surrounded minimally with respect to D_1

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† *A theorem concerning locally peripherally separable spaces*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 437–439.

and P by a finite collection Δ' of continua lying in D_1-D_3. By Theorem 32 there exists a sequence of simple domains $Q_1, Q_2, Q_3, \cdots$ bounded by simple closed curves $J_1, J_2, J_3, \cdots$ respectively, such that (1) for each n, Q_n contains $\overline{Q}_{n+1}$, and (2) if M is a closed and compact subset of $S-P$, there exists an integer n such that $\overline{Q}_n$ contains no point of M. Let C denote a component of $D_1-\overline{D}_2$, and let G denote the collection of all arcs T such that T crosses C and for some n is a subset of J_n. If T and T' are arcs of G, $\widehat{T}\cdot\widehat{T}'=0$. Since D_1 is simply connected but contains no simple domain containing P, for each n, J_n contains a point of $S-D_1$. Furthermore, let AP denote an arc lying in D_2 and in the component of D_1-P which contains C; then there exists an integer n_1 such that $\overline{Q}_{n_1}$ does not contain A; so if $n>n_1$, J_n contains a point of the segment $\widehat{AP}$ and a point of $S-D_1$, and therefore contains at least two arcs of G. However for each n, J_n contains at most a finite number of arcs of G. Hence G is countably infinite. Furthermore, suppose that L is an arc lying in C from one arc of G to another arc of G. There exists an integer n_2 such that if $n>n_2$, $\overline{Q}_n$ contains no point of L. Hence, between any two arcs of G there are only a finite number of arcs of G. Each arc of G is between some two arcs of G. For, if T is an arc of G, there exists an integer n such that J_n separates T from P. By Theorem 30, J_n contains two arcs which cross C such that T is between them in C.

Let α denote a well-ordered sequence whose elements are the elements of G. Let N_1 and N_0 denote the first two elements of α. Let N_2 denote the first element of α such that N_1 is between N_0 and N_2 in C. Let N_{-1} denote the first element of α such that N_0 is between N_{-1} and N_1 in C. This process may be continued. The sequences $N_1, N_2, N_3, \cdots, N_\omega, \cdots$ and $N_0, N_{-1}, N_{-2}, \cdots, N_{-\omega}, \cdots$ may or may not be simple sequences, but since each is countable, the first contains a simple countable subsequence $T_1, T_2, T_3, \cdots$ running through it and the second contains a simple countable subsequence $T_0, T_{-1}, T_{-2}, \cdots$ running through it, such that if T is any arc crossing C, there exists an integer n such that T is between T_n and T_{-n} in C, and such that the linear order of these arcs in C is the same as the order in the sequences.

Since C contains one and only one of the continua of Δ' and by Theorem 31 space is metric, there exists in C a connected domain C' such that (1) C' contains this element of Δ', and (2) $\overline{C}'$ is a subset of C. For each pair of consecutive integers (positive or negative), a and b with $a<b$, let U_a denote an arc irreducible from T_a to T_b lying in C', and let M_a denote the interval of T_a between the end points of U_a and U_{a-1} in T_a. The point set $\sum M_a$ is closed, since each M_a is for some n a closed subset of $C\cdot J_n$, for each n J_n contains only a finite number of the arcs of G, and the limiting set of $J_1, J_2, J_3, \cdots$ is

the point P. Also $\sum U_a$ is closed; for suppose that O is a limit point of the set $\sum U_a$ not belonging to it. The point O belongs to $\overline{C}'$ and therefore, to C. There exist a connected domain V lying in C and containing O but not intersecting more than one of the simple closed curves $J_1, J_2, J_3, \cdots$, and an infinite collection H of segments $\widehat{U}_a$ such that if $\widehat{U}_a$ is a segment of H, $\widehat{U}_a$ intersects V. Hence, $V+H^*$ is a connected subset of C not intersecting more than one of the arcs $T_1, T_2, T_3, \cdots, T_0, T_{-1}, T_{-2}, \cdots$, but having limit points in infinitely many of them. This is a contradiction. So $L=\sum(M_a+U_a)$ is closed and is, because of its method of construction, an open curve.

Now suppose that T is any arc crossing C. There exists an integer n such that T lies between T_n and T_{-n} in C. Hence T intersects that interval of L which is irreducible from T_n to T_{-n}. Furthermore, if this process is carried out in each component of $D_1-\overline{D}_2$ and Δ denotes the collection of open curves so obtained, Δ^* separates P from the boundary of D_1, and since no component of D_1-P contains more than one element of Δ', no component contains more than one element of Δ. Hence, Δ is minimal with respect to D_1 and P.

Let D denote the component of $S-\Delta^*$ which contains P. The domain D is pseudo-simple with respect to D_1 and P.

THEOREM 34. *If L is an element of a collection of open curves which surrounds a pseudo-simple domain D minimally with respect to a simply connected domain Q and a point P, and C is the component of $D-P$ whose boundary contains a point of L, then no ray of L separates C from the boundary of Q in $S-P$.*

Proof. Suppose the contrary is true. Then L contains a ray $\overrightarrow{B_1B_2}$ such that B_1 belongs to $\overline{C}$ and $\overrightarrow{B_1B_2}+P$ separates C from β, the boundary of Q. By Theorem 18, $C+P+B_1$ contains an arc PB_1 from P to B_1. The ray $\overrightarrow{PB_2}=PB_1+\overrightarrow{B_1B_2}$ separates $C-C\cdot PB_1$ from β, and by Theorem 20, $S-\overrightarrow{PB_2}$ is the sum of two connected domains I and E where I contains $C-C\cdot\overrightarrow{PB_2}$ and is a subset of Q and E contains β. Let E_1F_1 and E_2F_2 denote two arcs in C which abut on PB_1 from different sides. The points E_1 and E_2 both belong to $C-C\cdot PB_1$ and, hence, to I. Therefore I contains an arc E_1E_2 from E_1 to E_2, and $E_1E_2+E_1F_1+F_1F_2(\text{of } PB_1)+E_2F_2$ contains a simple closed curve lying in Q one of whose complementary domains lies in Q and contains P. This is a contradiction.

THEOREM 35. *If L is an element of a collection Δ of open curves which surrounds a pseudo-simple domain D minimally with respect to a simply connected domain Q and a point P, then no two segments lying in D abut on L from different sides.*[†]

[†] Two segments abut on a ray or open curve from different sides provided the ray or open curve contains an arc on which the segments abut from different sides.

Proof. Suppose, on the contrary, that D contains two mutually exclusive segments which abut on L from different sides. It follows, using Theorem 11(b), that there exist an arc A_1A_2 and two segments, $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$, all lying in the same component C of $D-P$, such that the segments $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$ abut on L from different sides and the point set $J=A_1B_1+B_1B_2(\text{of } L)+A_2B_2+A_1$ is a simple closed curve lying in Q. One of the complementary domains of J is a subset of Q. Hence, L contains two rays, $\overrightarrow{B_1B_2}$ and $\overrightarrow{B_2B_1}$, whose common part is the arc B_1B, one of which has the property that it together with P separates C from the boundary of Q. This is a contradiction of the preceding theorem.

THEOREM 36. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P, β is the boundary of D, and J is a simple closed curve lying in $\overline{D}$ such that $J\cdot\beta$ is connected, then one of the complementary domains of J is a subset of D.*

Proof. Let Δ denote a collection of open curves which surrounds D minimally with respect to Q and P, let L denote the element of Δ which contains $J\cdot\beta$, and let C denote the component of $D-P$ which contains $J\cdot(D-P)$. By Theorems 7 and 9, $J\cdot\beta+P$ does not separate space, and there exists an arc T from a point of $L-J\cdot L$ to a point on the boundary of Q which contains no point of J. One of the complementary domains I of J is a subset of Q. Therefore, I contains no point of T and, with the help of Theorem 35, it may be shown that I contains no point of L. But I does not contain P.† Hence, since I contains points of C, I is a subset of C and, consequently, of D.

THEOREM 37. *If D is a pseudo-simple domain, L is an element of a collection Δ of open curves which surrounds D minimally with respect to a simply connected domain Q and a point P, and OX and $O'X'$ are arcs lying in $S-\overline{D}$ except for their end points X and X' which lie on L, then the arcs OX and $O'X'$ abut on L from the same side.*

Proof. With the help of Theorems 11(b) and 34 it can be shown that there exists an arc FF' lying in D except for the end points F and F', which lie on L, such that the interval FF' of L contains X and X' as interior points. Then by Theorem 36, $J=FF'+FF'(\text{of } L)$ is a simple closed curved one of whose complementary domains is a subset of D. The other complementary domain of J contains the segments $\widehat{OX}$ and $\widehat{O'X'}$. Hence, the arcs approach FF' of L and, therefore, L from the same side.

THEOREM 38. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P, and OX and $O'X'$ are arcs which have their*

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† From the definition of a pseudo-simple domain, Q contains no simple domain containing P.

end points X and X' in an element L of a collection Δ of open curves which surrounds D, but have no other points in common with the boundary of D, and approach L from different sides, then one of these arcs lies in D and the other lies in $S-\overline{D}$.

Theorem 38 is a consequence of Theorems 35 and 37.

Theorem 39. *If Δ is a collection of open curves which surrounds a pseudo-simple domain minimally with respect to a simply connected domain Q and a point P, then the boundary of D is Δ^*, and every point of Δ^* is accessible from D.*

Proof. Let X denote a point of an open curve L of Δ, and let AXB denote an interval of Δ containing X as an interior point. By Theorem 19 there exist two segments, $\widehat{EX}$ and $\widehat{E'X}$, which abut on AXB at X from different sides. The segments, $\widehat{EX}$ and $\widehat{E'X}$, contain segments, $\widehat{FX}$ and $\widehat{F'X}$, respectively, which abut on L from different sides and do not contain P or any point of the boundary of D except X. By Theorem 38 one of the latter two segments is a subset of D.

Theorem 40. *If X is a point of the component L of the boundary β of a domain D which is pseudo-simple with respect to a simply connected domain Q and a point P, and M is a closed and compact point set not containing X, there exists a simple closed curve lying in $S-(D+M)$ whose common part with β is an interval of L containing X as an interior point, and whose interior with respect to P contains no point of M.*

Proof. Let XP denote an arc from X to P lying in $D+X$, and let J denote a simple closed curve separating X from P such that $J\cdot XP$ is a connected subset of D. The curve J contains a point of $S-D$. Let O denote the first point that XP has in common with J, and let AOB denote the interval of J containing O and lying in D except for its end points A and B which lie on L. The interval AB of L contains X because one of the complementary domains of the simple closed curve $AOB+AB$ is a subset of D and since it can not contain P, it must contain the segment $\widehat{OX}$ of XP. Let AZB denote the arc of J such that $J=AXB+AZB$. In the order A, Z, B, let A_0 denote the last point that AZB has in common with the ray $\overrightarrow{XA}$ of L, and let B_1 denote the next point that AZB has in common with β if it is not in L. Let L_1 denote the component of β containing B_1, and let A_1 denote the last point that AZB has in common with L_1. Let B_2 denote the next point that AZB has in common with β if it is not in L, let L_2 denote the open curve of β containing B_2, and let A_2 denote the last point that AZB has in common with L_2. This process can be continued (for a finite number of steps) until for some integer n the next point that AZB has in common with β following A_n is a point B_0 of

L. Since A_0 precedes B_0 in AZB, B_0 belongs to the ray $\overrightarrow{AB}$ of L. Let J_1 denote the simple closed curve A_0XB_0(of L)$+A_0B_1$(of AZB)$+A_1B_1$(of L_1)$+A_1B_2$(of AZB)$+ \cdots +A_{n-1}B_n$(of AZB)$+A_nB_n$(of L_n)$+A_nB_0$(of AZB). (See Figure 3.) It is clear that if B_1 exists, J_1 lies in $S-D$, because by Theorem 11(b) any

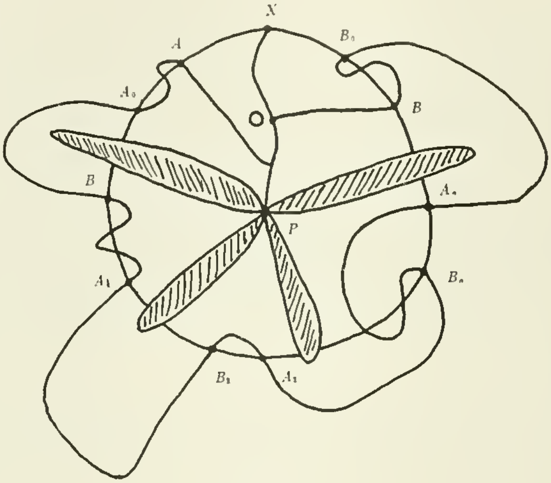

FIGURE 3

segment lying in D having its end points in different components of β contains P, and J_1 does not contain P. Now in case $J_1=A_0XB_0$(of L)$+A_0B_0$(of AZB), suppose that $\widehat{A_0B_0}$ lies in D. Then J_1 lies in $\bar{D}$ and $\beta\cdot J_1$ is connected; so by Theorem 36 one of the complementary domains I of J_1 is a subset of D. Let $P'X$ denote an arc lying in I except for X which is on the boundary of I. Now I does not contain P and J_1 does not intersect $\widehat{PX}+P$; hence, I does not contain $\widehat{PX}$, and therefore PX and $P'X$ abut on AXB from different sides. But since both $\widehat{PX}$ and $\widehat{P'X}$ lie in D, this contradicts Theorem 35.

Thus in either case J_1 is a subset of $S-D$. By Theorem 16 one of the complementary domains I_1 of J_1 is a subset of $S-\bar{D}$ and is the interior of J_1 with respect to P. Let X' denote a point of $J_1-J_1\cdot L$, and let J_2 denote a simple closed curve separating X from $P+M+X'$. Let I_2 denote the interior of J_2 with respect to P. By Theorem 11 of Chapter III of *Foundations*, J_1+J_2 contains a simple closed curve J_3 such that (1) I_3, the interior of J_3 with respect to P, is a subset both of I_1 and of I_2, and (2) $J_3\cdot L$ is an arc containing X as an interior point. Let $A'X'B'$ denote an arc lying in I_3 except for points A' and B' which belong to $J_3\cdot\widehat{A_0X}$ and $J_3\cdot\widehat{B_0X}$ respectively. Then $J_4=A'XB'$(of L)$+A'X'B'$ is a simple closed curve lying in $S-(D+M)$ whose interior with respect to P is a subset of I_3 and hence, contains no point of M. Furthermore, the common part of J_4 with β is the arc $A'XB'$ which is an interval of L containing X as an interior point.

THEOREM 41. *If D is a pseudo-simple domain, then $S-\bar{D}$ is connected.*

Proof. The boundary β of D is the sum of a finite number of mutually exclusive open curves $L_1, L_2, \cdots, L_j$. By Theorem 40 there exists, for each point X of β, a simple closed curve J_X such that J_X is a subset of $S-D$ and $J_X \cdot \beta$ is for some $n \leqq j$ an interval T_X of L_n containing X as an interior point. Let I_X denote the complementary domain of J_X which does not contain D. By Theorem 16, I_X is a subset of $S-\bar{D}$. Now for each n, $n \leqq j$, let $D_n = \sum I_X$ for X on L_n. All the segments $J_X - T_X$ for X on L_n lie in $S-\bar{D}$ and abut on L_n from the same side. Hence the domains I_X overlap, so that D_n is a connected subset of $S-\bar{D}$. Furthermore, if T is any arc in $S-D$ which abuts on L_n at a point X, it abuts on L_n from the same side as $J_X - T_X$, and hence, contains points of I_X and D_n. So for each $n \leqq j$, only one component of $S-\bar{D}$ has boundary points on L_n.

Now suppose that $S-\bar{D}$ is not connected. Then there exist two mutually exclusive domains H and K such that $S-\bar{D}=H+K$. Since space is connected, both H and K have boundary points in β, but for no n, $n \leqq j$, does L_n contain boundary points of both H and K. Hence $\beta \cdot \overline{H}$ and $\beta \cdot \overline{K}$ are mutually exclusive closed point sets. Let AB denote an arc in $S-P$ irreducible from $\beta \cdot \overline{H}$ to $\beta \cdot \overline{K}$. Now since each D_n, $n \leqq j$, belongs either to H or to K, $\beta = \beta \cdot \overline{H} + \beta \cdot \overline{K}$; so AB contains only two points A and B of β, and these points belong to different open curves of the collection $L_1, L_2, \cdots, L_j$. Therefore, since AB does not contain P, the segment AB lies in $S-\bar{D}$, and hence, contains a point of I_A and a point of I_B. But, since I_A and I_B belong to H and K respectively, this is a contradiction.

THEOREM 42. *Every pseudo-simple domain is strongly simply connected.*

This theorem is a consequence of Theorems 16 and 41.

DEFINITION. A domain D is said to be *internally simple* provided that for each point X of D there exists a simple domain containing X and lying together with its boundary in D.

THEOREM 43. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P, then each component of $D-P$ is internally simple.*

Proof. Let U denote a component of $D-P$, and let X denote a point of U. Let AP denote an arc from A to P lying except for P in U. By Theorems 2 and 10 there exists a simple closed curve J crossing XP. If J intersects the boundary of D, then J contains an arc AYB crossing XP and lying except for the points A and B of L in D. Let J_1 denote the simple closed curve AB(of L)$+AYB$. One of the complementary domains of J_1 is a subset of D and contains X. Let J_X denote a simple closed curve separating J_1+P from X. If J does not intersect the boundary of D, let J_X denote J. Then, one of the complementary domains I_X of J_X contains X, and I_X+J_X is a subset of U. Hence, U is internally simple.

THEOREM 44. *If D is a pseudo-simple domain, $S-\overline{D}$ is internally simple.*

Proof. The domain D is pseudo-simple with respect to a domain Q and a point P. Let Δ denote the collection of mutually exclusive open curves whose sum is the boundary of D, and let L denote an element of Δ. Let X denote a point of $S-\overline{D}$, and let AX denote an arc irreducible from L to X and containing no point of Δ^*-L. Suppose that $S-(\overline{D}+AX)$ is the sum of two mutually exclusive domains H and K. Because no arc separates space, both H and K have boundary points in Δ^*. Furthermore, because of Theorem 40 no element of $\Delta-L$ contains points of both $\Delta^*\cdot\overline{H}$ and $\Delta^*\cdot\overline{K}$, nor does either of the rays of L from A contain points of both $\Delta^*\cdot\overline{H}$ and $\Delta^*\cdot\overline{K}$. Let AP denote an arc in $D+A$ from A to P. Since $AX+AP$ does not separate space, there exists an arc T irreducible from $\Delta^*\cdot\overline{H}$ to $\Delta^*\cdot\overline{K}$ in $S-(AX+AP)$. Now the segment $\widehat{T}$ is a subset of $S-\overline{D}$, for if $\widehat{T}$ were a subset of D, it would have one end on one ray of L from A and the other end on the other ray from A without intersecting AP, which by Theorems 35 and 36 is impossible. But it is also impossible for $\widehat{T}$ to lie in $S-\overline{D}$, for in this case H and K would have boundary points in the same open curve of Δ. Hence, $S-(\overline{D}+AX)$ is connected.

Let EYF denote an arc lying except for Y in $S-(\overline{D}+AX)$ and crossing AX at the point Y. Since $S-(\overline{D}+AX)$ is connected, there exists an arc EZF lying in it from E to F. The sum of the arcs, EYF and EZF, contains a simple closed curve J crossing AX at the point Y whose interior with respect to P is a simple domain I containing X such that $I+J$ is a subset of $S-\overline{D}$.

THEOREM 45. *If D_1 and D_2 are pseudo-simple domains with respect to domains Q_1 and Q_2 and a point P respectively, Q_2 is a subset of D_1, and C is a component of $D_1-\overline{D}_2$, then C is an internally simple, strongly simply connected domain.*

Proof. The component C is strongly simply connected; for if a simple closed curve J is a subset of $\overline{C}$, then by Theorems 16 and 41 the interior I of J with respect to P is a subset of both D_1 and $S-\overline{D}_2$.

If X is a point of C, then by Theorems 43 and 44 there exist two simple domains, I_1 and I_2, each containing X such that $\overline{I}_1$ and $\overline{I}_2$ are subsets of D_1-P and $S-\overline{D}_2$ respectively. By Theorem 12 of Chapter III of *Foundations*, $I_1\cdot I_2$

contains a simple domain I_X which contains X and whose boundary is a subset of the sum of the boundaries of I_1 and I_2.[†] Therefore, $\bar{I}_X$ is a subset of C, and C is internally simple.

THEOREM 46. *If D is an internally simple, simply connected domain and AB is an arc lying in D except for its end points which belong to the boundary of D, then $D-D\cdot AB$ is the sum of two mutually exclusive, internally simple, simply connected domains, each having AB on its boundary.*

Proof. For each point X of $\widehat{AB}$ let I_X denote a simple domain containing X such that $\bar{I}_X$ is a subset of D. Let J_X denote the boundary of I_X and let C denote a simple closed curve separating X from $A+B+J_X$. By Theorem 10, $C+\widehat{AB}$ contains a simple closed curve C' separating X from $A+B+J_X$, whose common part with AB is the sum of two continua. Let $A'XB'$ denote the interval of AB containing X such that $A'XB'\cdot C'=A'+B'$. One of the complementary domains I' of C' is a subset of D and contains the segment $A'XB'$. The point set $I'-\widehat{A'XB'}$ is the sum of two simple domains I_{X1} and I_{X2}, one on one side of AB and the other on the other side of AB. Now if the subscripts 1 and 2 are so used so that all the domains I_{X1} for X on AB are on the same side of AB and all the domains I_{X2} are on the other side of AB, $D_1=\sum I_{X1}$ and $D_2=\sum I_{X2}$ are two connected domains which are subsets of D. Now each point O of $D-\widehat{AB}$ may be joined to a point of $\widehat{AB}$ by an arc OE lying in $D-\widehat{AB}$ except for its one end point E, and $\widehat{OE}$ must intersect either D_1 or D_2. If for each i, $i=1, 2$, H_i denotes the set of all points O such that the segment $\widehat{OE}$ intersects D_i, then H_1 and H_2 are connected domains and $H_1+H_2=D-\widehat{AB}$. But if D is simply connected, no arc segment lying in D can abut on AB from different sides since neither A nor B belongs to D. Hence H_1 and H_2 are mutually exclusive, and AB belongs to the boundary of each of them.

Now suppose that J is a simple closed curve lying in H_1. One of the complementary domains I of J is a subset of D. Therefore, I contains no point of AB. But I has limit points in H_1, namely, the points of J, and hence contains points of H_1. Consequently I is a subset of H_1, and therefore H_1 is simply connected. Likewise, H_2 is simply connected.

Let X denote a point of H_1. Since D is internally simple, there exists a simple domain I_X' containing X and lying together with its boundary J_X' in D. By Theorem 3 there exists a simple closed curve J_X separating X from $J_X'+AB$. The curve J_X is a subset of H_1, and hence one of its complementary domains I_X is a subset of H_1 and contains X. Consequently, H_1 is internally simple and likewise H_2 is internally simple.

[†] The notation here does not correspond to that of Theorem 12 of Chapter III of *Foundations*.

THEOREM 47. *If D is an internally simple, simply connected domain, β is the boundary of D, J is a simple closed curve lying in D, and AB is an arc irreducible from J to the boundary of D, then $D-(J+\widehat{AB})$ is the sum of two mutually exclusive, internally simple, simply connected domains, one having J for its boundary, and the other having for its boundary $J+AB+\beta$.*

Theorem 47 may be established by methods similar to those used in proving Theorem 46.

THEOREM 48. *If D is a pseudo-simple domain and AB is an arc lying in $S-\overline{D}$ except for the points A and B which lie on the boundary of D, then $S-(\overline{D}+AB)$ is the sum of two connected domains whose boundaries have as their common part the arc AB.*

Proof. From Theorems 16, 41, 44, and 46, $S-(\overline{D}+AB)$ is the sum of two connected domains whose boundaries contain AB in their common part. And if β denotes the boundary of D, it is clear from Theorems 38 and 40 that this common part cannot contain a point of $\beta-(A+B)$.

THEOREM 49. *If under the hypothesis of Theorem 45, AXB is an arc lying in C except for the points A and B which lie on the boundary of C, then $C-\widehat{AXB}$ is the sum of two connected domains whose boundaries have as their common part the arc AXB.*

Theorem 49 may be proved by an argument similar to that of the preceding theorem if we use Theorems 45 and 46 and a theorem similar to Theorem 40 and established in a similar manner.

DEFINITION. Suppose that X_1, X_2, X_3, and X_4 are points of β, the boundary of a pseudo-simple domain. Then X_1+X_3 is said to *ordinally separate* X_2 from X_4 on β if there exists a simple closed curve J whose common part with β is X_1+X_3 and which separates X_2 from X_4.

THEOREM 50. *Suppose that D is a pseudo-simple domain, β is the boundary of D, and X_1, X_2, X_3, and X_4 are four distinct points of β. Then one pair of them ordinally separates the other two on β; and if X_1+X_3 ordinally separates X_2 from X_4 on β, then* (1) *X_2+X_4 ordinally separates X_1 from X_3 on β,* (2) *every simple closed curve J containing a point of D and $S-D$ such that $J\cdot\beta=X_1+X_3$ separates X_2 from X_4, but* (3) *X_1+X_2 does not ordinally separate X_3 from X_4 on β.*

Proof. Let X_1AX_2 and X_1BX_2 denote arcs lying except for X_1 and X_2 in D and $S-\overline{D}$ respectively. Suppose that $J'=X_1AX_2+X_1BX_2$ does not separate X_3 from X_4. Now let AX_3 and BX_3 denote arcs lying except for X_3 in D and $S-\overline{D}$ respectively, such that $AX_3\cdot J'=A$ and $BX_3\cdot J'=B$. With respect to X_4, the interior of the simple closed curves formed by one pair of the three arcs AX_1B, AX_2B, and AX_3B contains the other arc segment. Hence, one pair of the points X_1, X_2, X_3, and X_4 ordinally separates the other two on β.

Suppose that X_1+X_3 ordinally separates X_2 from X_4 on β. Then there exists by a construction similar to that of the above paragraph two arcs X_2AX_4 and X_2BX_4 lying except for X_2 and X_4 in D and $S-\overline{D}$ respectively whose sum is a simple closed curve separating X_1 from X_3. If J is a simple closed curve containing a point of D and a point of $S-\overline{D}$ whose common part with β is X_1+X_3, then J is the sum of two arcs $X_1A'X_3$ and $X_1B'X_3$ lying except for X_1 and X_3 in D and $S-\overline{D}$ respectively. Let C denote the simple closed curve $X_2AX_4+X_2BX_4$. By a double application of Theorem 10, $C+\widehat{X_1A'X_3}+\widehat{X_1B'X_3}$ contains a simple closed curve C' whose common part with $\widehat{X_1A'X_3}$ is connected and whose common part with $\widehat{X_1B'X_3}$ is also connected and which separates X_2 from X_4. It is clear now that J separates X_2 from X_4.

Now suppose that X_1+X_2 ordinally separates X_3 from X_4 on β. Then there would exist a simple closed curve, having the properties of J in the preceding paragraph, which does not separate X_2 from X_4. This is impossible if X_1+X_3 ordinally separates X_2 from X_4.

Arguments of a similar nature show that if order on β is interpreted by the notion of ordinal separation, then we have

THEOREM 51. *If β is the boundary of a pseudo-simple domain, the points of β have a cyclic order which preserves the ordinary order on any open curve component of β.*

THEOREM 52. *Suppose that D_1 and D_2 are pseudo-simple domains with respect to domains Q_1 and Q_2 and a point P respectively, Q_2 is a subset of D_1, X_1, X_2, X_3, and X_4, and Y_1, Y_2, Y_3, and Y_4 are the end points of four mutually exclusive arcs T_1, T_2, T_3, and T_4 which cross $D_1-\overline{D}_2$. Then if X_1+X_3 ordinally separates X_2 from X_4 on β_1, the boundary of D_1, then Y_1+Y_2 ordinally separates Y_2 from Y_4 on β_2, the boundary of D_2, and conversely.*

Proof. Let X_1AX_2 denote an arc lying except for the points X_1 and X_3 in $S-\overline{D}_1$, and let Y_1BY_3 denote an arc lying except for the points Y_1 and Y_3 in D_2. Let J denote the simple closed curve $T_1+X_1AX_3+T_3+Y_1BY_3$. Since X_1+X_3 ordinally separates X_2 from X_4 on β_1, J separates X_2 from X_4. But J does not intersect T_2 or T_4 and hence, J separates Y_2 from Y_4. Therefore, Y_1+Y_2 ordinally separates Y_2 from Y_4 on β_2. Likewise, the converse is true.

2. CONSEQUENCES OF AXIOMS $0-4$, 5_1^* AND 7

AXIOM 7. *The set of all points is completely separable.*[†]

[†] This axiom is numbered 7 because it is so numbered in *Foundations*. See *Foundations*, p. 412.

The purpose of this section is to outline an argument showing that a space satisfying Axioms 0–4, 5_1^*, and 7 is a subset of a space satisfying Axioms 0–4, 5_1^*, and 7 which contains no edge points. Considerable detail will be left to the reader, and notation introduced in a definition, when used subsequently without explanation, will be understood to have the significance given it in the definition.

Definition. Suppose that P is an edge point. Suppose, furthermore, that there exist for each n, a pseudo-simple domain D_n containing $\overline{D}_{n+1}$ whose boundary β_n is the sum of a finite number of open curves, $L_{n1}, L_{n2}, \cdots, L_{nq_n}$, and for each n and q, $q \leqq q_n$, two sequences of mutually exclusive arcs, $T_{1,nq}, T_{2,nq}, \cdots$ and $T_{-1,nq}, T_{-2,nq}, \cdots$, crossing $D_n - \overline{D}_{n+1}$ such that, denoting $T_{m,nq} \cdot \beta_n$ and $T_{m,nq} \cdot \beta_{n+1}$ by $X_{m,nq}$ and $Y_{m,nq}$ respectively,†

(0) for each n, D_n is pseudo-simple with respect to P and a domain Q_n which is a subset of a region of G_n of Axiom 1, which is in turn a subset of D_{n-1}, $D_0 = S$;

(1) for each m, L_{nq} contains $X_{m,nq}$;

(2) for each n, the points $X_{1,n1}, X_{2,n1}, \cdots, \cdots, X_{-2,n2}, X_{-1,n2}, X_{1,n2}, X_{2,n2}, \cdots, \cdots, \cdots, X_{-2,nq_n}, X_{-1,nq_n}, X_{1,nq_n}, X_{2,nq_n}, \cdots, \cdots, X_{-2,n1}, X_{-1,n1}$ have on β_n the cyclic order indicated;‡

(3) for each $Y_{m,nq}$, there exists an integer r, $r \geqq q$, such that $X_{m,(n+1)r} = Y_{m,nq}$;

(4) for each n and q, $q \leqq q_n$, neither of the sequences, $T_{1,nq}, T_{2,nq}, \cdots$ and $T_{-1,nq}, T_{-2,nq}, \cdots$ has a sequential limit set. Let Ws denote the point set consisting of P together with all the sets L_{nq} and $T_{m,nq}$. Such a set, Ws, is said to be a *radial web skeleton* about P.

Theorem 53. *If R is a region containing an edge point P, R contains a radial web skeleton about P.*

Theorem 53 may be proved with the help of Theorems 15, 18, 32, 33, 45, and 52 and a method similar to that of Theorem 33.

Definition and notation. Suppose that Ws is a radial web skeleton about an edge point P. For each n and q, $q \leqq q_n$, L_{nq} of Ws is the sum of two rays L_{nq}^+ and L_{nq}^- having only one point in common and containing $X_{1,nq}, X_{2,nq}, \cdots$ and $X_{-1,nq}, X_{-2,nq}, \cdots$ respectively. Such rays are said to be *rays on* β_n and to have the cyclic order of the points $X_{m,nq}$. The point set Ws is said to connect L_{nq}^+ to $L_{(n+1)r}^+$ if $T_{1,nq}$ contains points of both of them. Likewise, Ws is said to connect L_{nq}^- to $L_{(n+1)r}^-$ if $T_{-1,nq}$ contains points of both of them.

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† When an integral subscript letter is followed by a comma, its range is both the positive and negative integers.

‡ This order is selected arbitrarily on β_1; then it is transferred from β_1 to β_2, from β_2, to $\beta_3, \cdots$ by the arcs $T_{m,nq}$ using Theorem 52.

THEOREM 54. *Suppose that Ws is a radial web skeleton about an edge point P, and Ws connects L^+_{nq} to $L^+_{(n+1)r}$ and $L^-_{nq'}$ to $L^-_{(n+1)r'}$. Then if $q'=q+1$, $r'=r+1$, and conversely.*

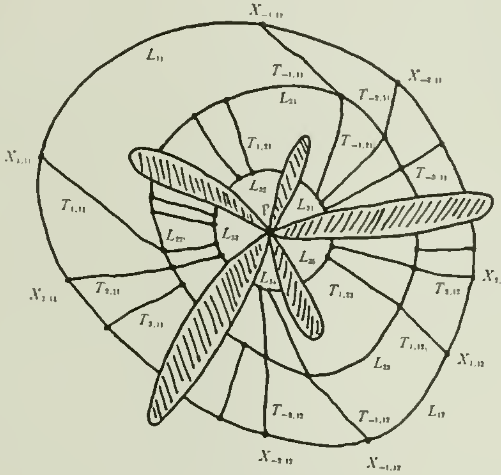

FIGURE 4

Proof. Suppose on the contrary, for instance, that $T_{1,11}$ contains a point of L^+_{11} and of L^+_{23} while $T_{-1,12}$ contains a point of L^-_{12} and of L^-_{25}. Since $T_{1,11}+T_{-1,12}+P$ does not separate space, there exists an arc T from L_{24} to β_1 of Ws lying except for its end points in $D_1-\bar{D}_2$. Let $T\cdot\beta_1=X$ and $T\cdot\beta_2=Y$. Since Y is between $Y_{1,11}$ and $Y_{-1,12}$ on β_2, X is between $X_{1,11}$ and $X_{-1,12}$ on β_1 by Theorem 52. Hence X is either on the subray of L^+_{11} from $X_{1,11}$ or on the subray of L^-_{12} from $X_{-1,12}$. Suppose the former, and a similar argument will apply to the other case. Since the sequence of arcs $T_{1,11}$, $T_{2,11}$, $T_{3,11}$, $\cdots$ has no sequential limit set, there is some one of them, $T_{m,11}$, which has no point in common with T such that $X_{m,11}$ is between X and $X_{-1,12}$ on β_1. But $Y_{m,11}$ belongs to L^+_{23} and is not between Y and $Y_{-1,12}$ on β_2, which contradicts Theorem 52.

A similar argument will show the converse to be true; that is, if $r'=r+1$, then $q'=q+1$.

DEFINITION. Suppose that Ws is a radial web skeleton. Then L^+_{nq} and $L^-_{n(q+1)}$ of Ws are said to be *adjacent rays* on β_n.†

THEOREM 55. *Suppose that Ws is a radial web skeleton. Then Ws connects each pair of adjacent rays on β_n to a pair of adjacent rays on β_{n+1}, and if Ws*

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† When $q=q_n$, $q+1$ is to be interpreted as 1.

connects either of a pair of adjacent rays on β_n to a ray on β_{n-1}, then Ws connects this pair to a pair of adjacent rays on β_{n-1}.

Notation. If Ws is a radial web skeleton, let C_{nq} denote the component of $D_n-\bar{D}_{n+1}$ whose boundary β_{nq} contains L_{nq} of Ws.

THEOREM 56. *If Ws is a radial web skeleton, then $\beta_{nq}\cdot\beta_{n+1}$ is the sum of a finite collection H_{nq} of consecutive rays on β_{n+1} consisting of two extreme rays, $L^-_{(n+1)q}$ and $L^+_{(n+1)q'}$, $q\leqq q'$, and all other rays on β_{n+1} between $L^-_{(n+1)q}$ and $L^+_{(n+1)q'}$; and $L^-_{(n+1)q}$ and $L^+_{(n+1)q'}$ are the only rays of H_{nq} which are connected by Ws to rays of β_n.*

THEOREM 57. *If Ws is a radial web skeleton and n and q are integers such that neither L^+_{nq} nor $L^-_{n(q+1)}$ of Ws is connected to a ray on β_{n-1} by Ws, then there exists a sequence of mutually exclusive arcs $U_{1[nq]}$, $U_{2[nq]}$, $U_{3[nq]}$, $\cdots$ † such that* (1) *for each m, $m>0$, $U_{m[nq]}$ is an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these two points in $D_{n-1}-\bar{D}_n$, $D_0=S$, and* (2) *the sequence has no sequential limit set.*

Proof. Let C denote the component of $D_{n-1}-\bar{D}_n$ whose boundary contains $L^+_{nq}+L^-_{n(q+1)}$, and let L denote $\bar{C}\cdot\beta_{n-1}$.‡ Let G denote the collection of all simple domains which lie together with their boundaries in C. By Theorems 44 and 45, G covers C. Since space is completely separable, G contains a countable subcollection $I_1, I_2, I_3, \cdots$ covering C. For each integer m, $m>0$, let M_m denote an arc irreducible from J_m, the boundary of I_m, to L lying except for one end point in C. By Theorem 47, $C-C\cdot(J_m+M_m)$ is the sum of two connected domains, I_m and E_m, where the boundary of E_m consists of J_m+M_m together with the boundary of C. For each m, let $U'_{m[nq]}$ denote an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these end points in E_m. By Theorem 49, $U'_{m[nq]}$ separates C into two internally simple, simply connected domains. One of these domains V_m' has for its boundary $U'_{m[nq]}$ plus the subray of L^+_{nq} from $X_{m,nq}$ and the subray of $L^-_{n(q+1)}$ from $X_{-m,n(q+1)}$, and contains no point of I_m. If the arcs $U'_{m[nq]}$ are mutually exclusive, then they form the sequence $U_{1[nq]}$, $U_{2[nq]}$, $U_{3[nq]}$, $\cdots$. If this is not the case, let $U_{1[nq]}=U'_{1[nq]}$ and $V_1=V_1'$, and suppose that $U_{1[nq]}\cdot U'_{2[nq]}$ is not vacuous. Let A_1B_1 and A_2B_2 denote intervals of $U_{1[nq]}$ and $U'_{2[nq]}$ respectively which lie in C and contain $U_{1[nq]}\cdot U'_{2[nq]}$. By Theorems 10 and 45 and Theorem 13 of Chapter III of *Foundations*, there exists a simple closed curve J lying in C and separating $A_1B_1+A_2B_2$ from $X_{2,nq}+X_{-2,n(q+1)}$ such that the common part of J with each of the intervals $A_2X_{2,nq}$ and $B_2X_{-2,n(q+1)}$ of $U'_{2[nq]}$ is connected. Now since

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† This notation, $[nq]$, is used to restrict n and q to values for which the hypothesis of Theorem 57 is satisfied.

‡ When $n=1$, $\bar{C}\cdot\beta_{n-1}$ is to be interpreted as a point of L^-_{nq}.

$\widehat{A_2X}_{2,nq}$ and $\widehat{B_2X}_{-2,n(q+1)}$ are subsets of V_1 and contain no points of V_2', J contains an arc AB lying in $V_1\cdot V_2'$ except for A and B irreducible from $A_2X_{2,nq}$ to $B_2X_{-2,n(q+1)}$. Let $U_{2[nq]}=AB+AX_{2,nq}$ of $A_2X_{2,nq}+BX_{-2,n(q+1)}$ of $B_2X_{-2,n(q+1)}$, and let V_2 denote the component of $V_1-\widehat{U}_{2[nq]}$ which has the subray of L_{nq}^+ from $X_{2,nq}$ and the subray of $L_{n(q+1)}^-$ from $X_{-2,n(q+1)}$ on its boundary. The domain V_2 is a subset of $V_1\cdot V_2'$. This process may be continued and the required sequence of arcs constructed, for it is clear that since $V_1\supset V_2\supset V_3\supset\cdots$ and $\overline{V}_1\cdot\overline{V}_2\cdot\overline{V}_3\cdot\cdots=0$, the sequence of arcs has no sequential limit set.

Notation and construction. Now on a radial web skeleton it is possible to construct what will be called a *radial web*. Let Ws denote a radial web skeleton about an edge point P. For each integer triplet, m (positive or negative), $n>0$, and $0<q\leqq q_n$, there exists a continuous reversible transformation of the point set $T_{m,nq}-Y_{m,nq}$ of Ws into the number set $1/n\geqq t>1/(n+1)$.

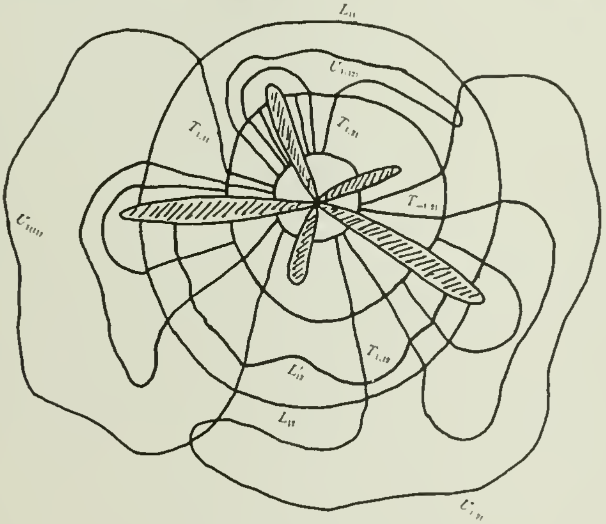

FIGURE 5

Let $X_{m,nq}^t$ denote the point of $T_{m,nq}$ whose transform is the number t. Then, of course, $X_{m,nq}=X_{m,nq}^{1/n}$. Let ρ denote a countable set of numbers everywhere dense on $0<t<1$ and containing all the reciprocals of positive integers except 1. For each pair of positive integers, n and q, $q\leqq q_n$, and for each t of ρ such that $1/n\geqq t>1/(n+1)$, let L_{nq}^t denote an open curve lying in C_{nq} such that (1) $L_{nq}^t=L_{nq}$ if $t=1/n$, (2) L_{nq}^t intersects $T_{m,nq}$ in the point $X_{m,nq}^t$, and (3) if t and t' are different numbers of ρ, $L_{nq}^t\cdot L_{nq}^{t'}=0$. Since ρ is countable, these open curves may be constructed one at a time by the use of Theorem 18 and repeated applications of Theorem 46.

For each pair of integers, n and q, $n>0$ and $q\leqq q_n$, such that L_{nq}^+ is not connected by Ws to a ray on β_{n-1}, there exists by Theorem 57 a sequence $U_{1[nq]}, U_{2[nq]}, U_{3[nq]}, \cdots$ of mutually exclusive arcs such that (1) for each m, $m>0$, $U_{m[nq]}$ is an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these points in $D_{n-1}-\bar{D}_n$, $D_0=S$, and (2) the sequence has no sequential limit set. For each t of ρ, let $\beta^t=\sum_{q=1}^{q_n} L_{nq}^t$. The point set W consisting of Ws together with all the point sets β^t and $U_{m[nq]}$ is called a *radial web* about P, and Ws is said to be the skeleton of W.

Notation and definition. If $X_{m,nq}$ is a point of a ray on β_n of the skeleton Ws of the radial web W which is not connected to a ray on β_{n-1} by Ws, let $N_{m,[nq]}$ denote $P+\sum_{r=n}^{\infty} T_{m,rp_r}$, where $p_n=q$ and for $r>n$, p_r is the integer such that $X_{m,rp_r}=Y_{m,(r-1)p_{r-1}}$. For each $[nq]$ and m, $m>0$, let $J_{m[nq]}$ denote the simple closed curve $N_{m,[nq]}+U_{m[nq]}+N_{-m,[n(q+1)]}$, and let $I_{m[nq]}$ denote the complementary domain of $J_{m[nq]}$ which contains $J_{m+1[nq]}-P$.† For each t of ρ, let $U_{m[nq]}^t$ denote the interval of $J_{m[nq]}-P$ which is irreducible from one component of β^t to another component of β^t with $t\leqq 1/n$.

For each pair of values, t' and t'' of ρ, $t'>t''$, let $\beta_{m,[nq]}^{t't''}$ denote the point set consisting of the interval of $N_{m,[nq]}$ between $\beta^{t'}$ and $\beta^{t''}$, when both t' and t'' are less than $1/n$, together with the rays of $\beta^{t'}$ and $\beta^{t''}$ from the end points of this interval which intersect $N_{m+1,[nq]}$. It is clear that $\beta_{m,[nq]}^{t't''}$ separates space into two mutually exclusive connected domains, one of which, $I_{m,[nq']}^{t't''}$, is a subset of $I_{m[nq]}$, is the sum of an infinite number of simple domains, and has $\beta_{m,[nq]}^{t't''}$ for its complete boundary. For each m, $m>0$, let $R_{m[nq]}^{t't''}$ denote $I_{m,[nq]}^{t't''}+I_{-m,[n(q+1)]}^{t't''}$. The boundary, $\beta_{m[nq]}^{t't''}$, of $R_{m[nq]}^{t't''}$ is $\beta_{m,[nq]}^{t't''}+\beta_{-m,[n(q+1)]}^{t't''}$. Let $P_{[nq]}^t$ denote the collection of all domains $R_{m[nq]}^{t't''}$, where $t''<t<t'$. The collection $P_{[nq]}^t$ will be hereinafter referred to as an *ideal point*.‡ The set of ideal points, $\sum_t P_{[nq]}^t$, will be called the *spur* $N_{[nq]}$ and $\sum N_{[nq]}$ will be called the *cluster* at P or about P.

THEOREM 58. *For each* $[nq]$, *the sequence* $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \cdots$ *has* P *for its sequential limit set and* $I_{1[nq]}\cdot I_{2[nq]}\cdot I_{3[nq]}, \cdots =0$.

THEOREM 59. *For each* m, $W-W\cdot\sum_{[nq]} I_{m[nq]}$ *is a closed and compact point set.*

THEOREM 60. *If* $P_{[nq]}^t$ *is an ideal point and* M *is a compact point set, there exists a number* δ *and an integer* $\bar{m}$, *such that if* $0<t'-t''<\delta$, $m>\bar{m}$, *and* $R_{m[nq]}^{t't''}$ *belongs to* $P_{[nq]}^t$, *then* $\bar{R}_{m[nq]}^{t't''}$ *contains no point of* M.

THEOREM 61. *If* t *is a number of* ρ *and* M *is a compact point set, there exists a number* $\bar{m}$ *such that if* $m>\bar{m}$, *then for no* $[nq]$ *does* $U_{m[nq]}^t$ *contain a point of* M.

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† The point sets $U_{m[nq]}$ may intersect the sets β^t in more than one point.

‡ This is analogous to a method used by R. L. Moore to define point in terms of *piece*. See R. L. Moore, *A set of axioms for plane analysis situs*, Fundamenta Mathematicae, vol. 25 (1935), pp. 13–28.

Notation and construction. By Theorem 24 the set of edge points may be put into a simple countable sequence $P_1, P_2, P_3, \cdots$. Let $W_1, W_2, W_3, \cdots$ and, for each integer j, $R_{1j}, R_{2j}, R_{3j}, \cdots$ and $D_{1j}, D_{2j}, D_{3j}, \cdots$ denote sequences of point sets such that (1) for each n and j, R_{nj} is a region of G_{n+j} of Axiom 1, (2) for each n and j, D_{nj} is a pseudo-simple domain with respect to P_j and a simply connected domain Q_{nj} lying with its boundary in R_{nj}, and D_{nj} contains $R_{(n+1)j}$, (3) for each j, W_j is a radial web about P_j whose skeleton is constructed on the domains, $D_{1j}, D_{2j}, D_{3j}, \cdots$, and lies in R_{1j}, (4) if $j'>j$ and $R_{1j'}$ contains a point of D_{1j}, then $\overline{R}_{1j'}$ is a subset of R_{1j}, and (5) for each integer j', there exists an integer m such that if $j<j'$, then $\overline{R}_{1j'}$ contains no point of $\overline{\sum_{[nq]} I_{m[nq]}}$ of W_j. Let G denote a countable collection of simple domains such that if X is a simple point of S and R is a region containing X, some domain of G contains X and lies together with its boundary in R. For each integer i, let H_i denote the collection of all the domains D such that either (1′) D is a simple domain of G which is a subset of a region of G_i of Axiom 1, or (2′) for some pair of integers n and j, $n+j \geqq i$, D is a pseudo-simple domain D_{nj}, or (3′) for some triplet of integers, m, n, and q, and some pair of numbers, t' and t'', of ρ, with $m \geqq i$ and $0<t'-t'' \leqq 1/i$, $D=R^{t't''}_{m[nq]}$ of W_j for some j.

Definition. If for some i and j, D is a domain of H_i and $P^t_{[nq]}$ is an ideal point defined from W_j, then D is said to *inclose* $P^t_{[nq]}$ provided that some domain of $P^t_{[nq]}$ together with its boundary is a subset of D.

Definition. Let S' be a space in which *point* is to be interpreted as meaning either a point of S or one of the ideal points defined by W_j for some j, and in which *region* is to be interpreted as meaning a point set g' consisting of the points of some domain D of H_i, for some i, together with the ideal points which D incloses. A point of S' which is also a point of S will be called an "ordinary" point and a point of S' which is not a point of S will be called an "ideal" point. Regions in S' will be referred to as *G'-regions.*

From here on the space to be considered is S'.

Theorem 63. *In S, point and limit point are unchanged by the above definition of point and region in S'.*

Theorem 64. *Every point of S' is a limit point of S.*

Theorem 65. *If g' is a G'-region, then $\bar{g}' = \overline{g' \cdot S}$.*

Theorem 66. *If J is a simple closed curve of "ordinary" points, then $S'-J$ is the sum of two mutually exclusive connected domains each having J for its complete boundary.*

Proof. The point set $S-J$ is the sum of two mutually separate connected sets H and K. Let $H'=\overline{H}-\overline{H}\cdot J$ and $K'=\overline{K}-\overline{K}\cdot J$. Suppose that H' has a point A in common with K'. The point A is an "ideal" point $P^{t}_{[nq]}$. By Theorem 60 there exists a number $\overline{m}_1$ such that if $m>\overline{m}_1$ and $R^{t't''}_{m[nq]}$ belongs to $P^{t}_{[nq]}$, then $\overline{R}^{t't''}_{m[nq]}\cdot S$ contains no point of J. But $R^{t't''}_{m[nq]}$ contains points of both H and K. Hence, $I^{t't''}_{m,[nq]}+\beta^{t't''}_{m,[nq]}$ is a subset of H, say, and $I^{t't''}_{-m,[n(q+1)]}+\beta^{t't''}_{-m,[n(q+1)]}$ is a subset of K. By Theorem 61 there exists a number $\overline{m}_2$ such that if $m>\overline{m}_2$, then $U^{t}_{m[nq]}$ does not intersect J. But for each m, $U^{t}_{m[nq]}$ intersects both $\beta^{t't''}_{m,[nq]}$ and $\beta^{t't''}_{-m,[n(q+1)]}$, $t''<t<t'$, which leads to a contradiction when m is greater than both $\overline{m}_1$ and $\overline{m}_2$. Hence, H' and K' have no point in common. Since H and K are connected and have J on their respective boundaries, H' and K' are connected and have J for their respective boundaries. By Theorem 64, $S'=J+H'+K'$.

THEOREM 67. *If* $P^{t}_{[nq]}$ *belongs to the cluster of "ideal" points about the point* P_j, *and* g' *is a* G'*-region containing the point* $P^{t}_{[nq]}$ *such that* $g'\cdot S$ *is not a domain of* $P^{t}_{[nq]}$, *then* $\overline{g}'\cdot S$ *contains* P_j.

Proof. Suppose that $g'\cdot S$ is not a domain of $P^{t}_{[nq]}$ but contains a domain R of $P^{t}_{[nq]}$ and $\overline{g}'\cdot S$ does not contain P_j.

Case I. If $g'\cdot S$ is of type $(1')$,† its boundary J separates R from P_j. Therefore, infinitely many of the simple closed curves $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \cdots$ of W_j would intersect J. But since J is a compact subset of S, the sequential limit set of $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \cdots$ of W_j would contain points of S other than P_j, contrary to Theorem 58.

Case II. If $g'\cdot S$ is of type $(2')$ and its boundary was used in the construction of the web $W_{j'}$, then j' is not j and one may see, from Theorem 58 and a method similar to that of Theorem 40, that the boundary of $g'\cdot S$ and a finite number of the arcs of the type $U^{t}_{m[nq]}$ of $W_{j'}$ contains a simple closed curve J lying in S and separating $g'\cdot S$, and hence R, from P_j. This leads to a contradiction as in Case I.

Case III. If $g'\cdot S$ is of type $(3')$, there exists an integer j', such that j' is not j and $g'\cdot S$ was defined using the web $W_{j'}$. Hence, $P_{j'}$ is not P_j and the boundary of $g'\cdot S$ together with one of the simple closed curves of the type $J_{m[nq]}$ of $W_{j'}$ contains a simple closed curve J lying in S and separating $g'\cdot S$, and hence R, from P_j. This leads to a contradiction as in Case I.

DEFINITION. For each i, let G'_i denote the collection of all G'-regions g' of S' such that $D=g'\cdot S$ is an element of H_i. The region g' is said to be *defined by* or *obtained from* D.

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† See Notation and construction, p. 87.

THEOREM 68. *If* $P^t_{[nq]}$ *is an "ideal" point, there exists an integer* k *such that no region of* G_k' *contains* $P^t_{[nq]}$ *except those regions of* G_k' *defined from domains belonging to* $P^t_{[nq]}$.

Proof. Let $P_{\bar{j}}$ denote the edge point P such that the cluster about P contains $P^t_{[nq]}$. Let $\bar{n}$ denote an integer such that $t > 1/\bar{n}$. There exists an integer $\bar{m}$ such that if i is an integer greater than $\bar{m}$ and R is a region in S of G_i of Axiom 1 which contains $P_{\bar{j}}$, then R is a subset of $D_{\bar{n}\bar{j}}$ in S. Suppose that g' is a region of G_i', $i > \bar{m}$, of the type (1′) or (2′) which contains $P^t_{[nq]}$.† Then by Theorem 67, $\bar{g}' \cdot S$ contains $P_{\bar{j}}$. But $\bar{g}' \cdot S$ is a subset of a region R of G_1 of Axiom 1. Hence, R contains $P_{\bar{j}}$ and points not belonging to $D_{\bar{n}\bar{j}}$, which is a contradiction. Likewise, if $j \geqq \bar{m}$ and $j \neq \bar{j}$, no region of type (3′) obtained from a domain of an "ideal" point of the cluster about P_j contains $P^t_{[nq]}$ because each such domain together with its boundary in S is a subset of a region of G_i of Axiom 1 with $i > \bar{m}$. Now by Theorem 58, for each integer j, $j < \bar{m}$ and $j \neq \bar{j}$, there exists an integer $\bar{m}_j$ such that for $m > \bar{m}_j$, $\overline{\sum_{[nq]} I_{m[nq]}}$ defined from W_j does not contain $P_{\bar{j}}$. Then if g' is a region of type (3′) of G_i', $i > \bar{m}_j$, defined by W_j, $\bar{g}' \cdot S$ does not contain $P_{\bar{j}}$. Hence, by Theorem 67, R does not contain $P^t_{[nq]}$. Let k denote an integer greater than $\bar{m}$ and for each j, $j < \bar{m}$ and $j \neq \bar{j}$, greater than $\bar{m}_j$. Thus, if $i > k$ and g' is a region of G_i' which contains $P^t_{[nq]}$, then g' was obtained from a domain of $P^t_{[nq]}$, for no two regions of type (3′) defined by $W_{\bar{j}}$ containing points of different spurs of the cluster about $P_{\bar{j}}$ have a point in common.

THEOREM 69. *The space* S' *satisfies Axiom* 1.

Proof. It can be shown with the help of Theorems 67 and 68 that S' satisfies parts (1), (2), and (3) of Axiom 1.‡ Considerable argument is required but is left to the reader.

It will now be shown that part (4) of Axiom 1 is satisfied. Suppose that $M_1, M_2, M_3, \cdots$ is a sequence of closed subsets of S' such that for each i, M_i contains M_{i+1} and is a subset of a region g_i' of G_i'.

Case I. Suppose that for infinitely many values of i, g_i' is of type (1′) or (2′). Suppose for simplicity that this is true for all integers i. Then for each i, $K_i = (\bar{g}_1' \cdot S) \cdot (\bar{g}_2' \cdot S) \cdots (\bar{g}_i' \cdot S)$ is a non-vacuous point set closed with respect to S such that K_i contains K_{i+1}. But for each i, $\bar{g}_i' \cdot S$, and hence K_i, is a subset of a region R_i of G_i of Axiom 1. By Axiom 1 for S, there exists a point P of S which for each i belongs to K_i. Hence in S', P belongs to the sequential limit set of M_1, M_2, M_3, $\cdots$ and therefore for each i, P belongs to M_i.

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† See **Notation and construction**, p. 87.

‡ See footnote p. 54. The role of the collections $G_1, G_2, G_3, \cdots$ is now to be taken by the collections $G_1', G_2', G_3', \cdots$.

Case II. If Case I does not occur, then there exists an integer $\bar{j}$ such that for infinitely many values of i, g_i' is a G'-region of type (3′) defined by the web $W_{\bar{j}}$; for otherwise infinitely many of these regions of type (3′) could be replaced by regions of type (2′) satisfying the above conditions. Suppose for simplicity that for each i, g_i' is of type (3′) and defined by $W_{\bar{j}}$. Since no two regions of type (3′) defined by $W_{\bar{j}}$ which have a point in common contain points of different spurs of the cluster about $P_{\bar{j}}$, there exist for each i, integers m, n, and q, and numbers t' and t'', $0<t''<t'<1$, such that (1) $R^{t't''}_{m[nq]}$ defined from $W_{\bar{j}}$ is $g_i'\cdot S$, (2) $[nq]$ does not vary with i, and (3) $m\to\infty$ and $(t'-t'')\to 0$, as $i\to\infty$. Since for each i, g_i' contains points of g_1', there exists a number t, $0<t<1$, such that as $i\to\infty$, $t'\to t$ and $t''\to t$. Hence $P^t_{[nq]}$ is common to M_1, M_2, M_3, $\cdots$.

Theorem 70. *Every G'-region is a simple domain.*

Theorem 71. *The space S' satisfies Axioms* 2, 3, 5_1^*, *and* 7 *and contains no edge points.*

Theorem 71 is a consequence of Theorems 64, 69, and 70 and the fact that the collection of all G'-regions is countable.

Theorem 72. *If J is a simple closed curve of "ordinary" points, I is one of its complementary domains, the points A and B separate the points C and D on J, and AXB is an arc of "ordinary" points such that $\widehat{AXB}$ is a subset of I, then $I-\widehat{AXB}$ is the sum of two simple domains I_1 and I_2 such that the boundary of I_1 is the simple closed curve ACB(of J)$+AXB$ and the boundary of I_2 is the simple closed curve ADB(of J)$+AXB$.*

Proof. Since the theorem holds for S, it can be shown to hold for S' with the help of Theorem 66.

Theorem 73. *Theorem* 72 *remains true when the segment AXB contains just one "ideal" point.*

Proof. Suppose for simplicity of notation that X is the "ideal" point of the segment AXB. Let W_j denote the web used in the definition of the domains of X, and let $X_{[nq]}$ denote the spur which contains X. A number of cases occur depending upon where the edge point P_j is. Only one of these cases will be treated. The others may be dealt with in a somewhat similar but more complicated manner.

Suppose that P_j does not belong to $J+AXB$. Let $J_{1[nq]}$, $J_{2[nq]}$, $J_{3[nq]}$, $\cdots$ denote the sequence of simple closed curves in W_j which has as its sequential limit set $P_j+X_{[nq]}$. It can be shown by using Theorem 10 that there exists a sequence $C_1, C_2, C_3, \cdots$ of simple closed curves whose sequential limit set is $P_j+N_{[nq]}$ such that for each k, (1) there is an integer $m, m\geqq k$, so that C_k is a subset of $J_{m[nq]}+AXB$, (2) $C_k\cdot\widehat{AX}$ is an arc $A_k'A_k''$ and $C_k\cdot\widehat{BX}$ is an arc $B_k'B_k''$, (3) the components of $C_k-C_k\cdot AXB$ are two segments, $A_k'X_kB_k'$ and $A_k''Y_kB_k''$, which approach the rays $\overrightarrow{AX}$ and $\overrightarrow{BX}$ in X from different sides but $A_k'X_kB_k'$ and $A_{k+1}'X_{k+1}B_{k+1}'$ approach these rays from the same side, and (4) the common part of C_k and C_{k+1} is at most P_j, C_k separating $C_{k+1}-C_{k+1}\cdot C_k$ from J. For each k, let I_{1k} and I_{2k} denote respectively the complementary domains of AA_k'(of AX)$+A_k'X_kB_k'+BB_k'$(of BX)$+ACB$ or AA_k''(of AX)$+A_k''Y_kB_k''+BB_k''$(of BX)$+ACB$ and AA_k'(of AX) $+A_k'X_kB_k'+BB_k'$(of BX)$+ADB$ or $AA_k''+A_k''Y_kB_k''+BB_k''$(of BX) $+ADB$ which are subsets of I. Let I_1 denote $I_{11}+I_{12}+I_{13}+\cdots$ together with the limit points of this set which do not belong to $J+AXB$, and let I_2 denote $I_{21}+I_{22}+I_{23}+\cdots$ together with the limit points of this set which do not belong to $J+AXB$. These point sets, I_1 and I_2, can be shown to be domains satisfying the conclusions of Theorem 72.

THEOREM 74. *Theorem 72 remains true if the arc AXB contains only a finite number of "ideal" points.*

Proof. The method of proof will be that of mathematical induction. Theorem 73 shows that Theorem 74 holds true when the number of "ideal" points is one. We shall suppose that the theorem holds for $j-1$ or less points and show that it holds for j points. Suppose that $AXB\cdot(S'-S)=Z_1+Z_2+\cdots+Z_j$. Let $N_{[nq]}$ denote a spur containing one of these "ideal" points, and let Z_r denote that one of them on $N_{[nq]}$ which is farthest in $N_{[nq]}$ from its end point. Now the argument of Theorem 73 may be applied, Z_r playing the role of X.

THEOREM 75. *If J is a simple closed curve containing only a finite number of "ideal" points, then $S'-J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

Proof. Let O denote an ordinary simple point of J, and let R denote a simple domain whose boundary contains no "ideal" points such that $R\cdot J$ is a subset of an arc in J of "ordinary" points. Let ACB denote an arc of "ordinary" points in R which has only the two points A and B in common with J and such that an arc ADB of J is a subset of R. It is now easy to see that, with the help of Theorem 74, Theorem 75 may be established.

THEOREM 76. *Theorem 72 remains true regardless of the composition of the arc AXB.*

Proof. By slight modifications of the arguments of R. L. Moore a number of the theorems of Chapter III of *Foundations* can be shown to hold true in S' if the simple closed curves involved contain at most a finite number of "ideal" points.† In particular Theorem 13 of Chapter III of *Foundations* holds true in S' for simple closed curves which contain at most a finite number of "ideal" points. Theorem 76 may then be established in the following manner.

Since the boundaries of G'-regions are simple closed curves containing at most a finite number of "ideal" points, it is easy to see that there exist two sequences of mutually exclusive arcs $A_1'X_1B_1'$, $A_2'X_2B_2'$, $A_3'X_3B_3'$, $\cdots$ and $A_1''Y_1B_1''$, $A_2''Y_2B_2''$, $A_3''Y_3B_3''$, $\cdots$ such that (1) for each k the segments $A_k'X_kB_k'$ and $A_k''Y_kB_k''$ are subsets of I, contain at most a finite number of "ideal" points and the points A_k', B_k', A_k'', and B_k'' belong to segments AC, BC, AD, and BD of J respectively, (2) for each k, the complementary domain of $A_k'X_kB_k'+A_k'AA_k''$ (of J)$+B_k'BB_k''$ (of$J$)$+A_k''Y_kB_k''$ which is a subset of I contains the segment of AXB and the corresponding complementary domain of $A_{k+1}'X_{k+1}B_{k+1}'+A_{k+1}'AA_{k+1}''$(of J)$+B_{k+1}'BB_{k+1}''$(of J) $+A_{k+1}''Y_{k+1}B_{k+1}''$, and (3) the sequential limit set of either of these two sequences of arcs is AXB. If for each k, I_{1k} and I_{2k} denote respectively the complementary domains of $A_k'CB_k'$(of J)$+A_k'XB_k'$ and $A_k''DB_k''$(of J) $+A_k''YB_k''$ which are subsets of I, then $I_1=\sum I_{1k}$ and $I_2=\sum I_{2k}$ satisfy the conclusions of Theorem 72.

Now as in Theorem 75 it is possible to establish

THEOREM 77. *If J is a simple closed curve containing an arc of "ordinary" points, then $S'-J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

With the help of Theorem 77 and methods similar to those in *Foundations* it can be shown that Theorem 74 remains true when the simple closed curve J is of any nature, provided it contains an arc of "ordinary" points. With this in mind, it is easy to see that Theorem 76 remains true when J is of any nature, provided it contains an arc of "ordinary" points. Then as in Theorem 74 it is possible to establish

THEOREM 78. *If J is a simple closed curve, $S'-J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

THEOREM 79. *The space S' satisfies Axiom 4.*

THEOREM 80. *The space S is homeomorphic with a subset of a space S' which satisfies Axioms 0–4, 5_1^* and 7 and contains no edge points.*

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† It may be helpful for the reader to note that an arc in S' may be approximated by an arc in S' which contains at most a finite number of "ideal" points since a G'-region has at most two "ideal" points in its boundary.

THEOREM 81. *The space S is homeomorphic with a subset of a plane or a sphere.*

Proof. J. H. Roberts[†] has shown that S' of Theorem 80 is homeomorphic with a subset of a plane or a sphere.

Some of the preceding results will now be summarized for metric spaces

THEOREM 82. *A locally connected, complete metric space in which the Jordan Curve Theorem*[‡] *is satisfied non-vacuously and in which Axiom 5_1^* holds true, is a cyclicly connected subset of a plane (if not compact) or a sphere (if compact).*

Proof. A locally connected complete metric space satisfies Axioms 0, 1, and 2. And such a space in which Axiom 5_1^* holds true satisfies Axioms 4 and 3 if the Jordan Curve Theorem holds true non-vacuously (Theorem B). Axiom 7 holds true in such a space by Theorem 31. Thus Theorem 82 follows from Theorem 81.

THEOREM 83. *A locally connected, locally peripherally connected, complete metric space in which the Jordan Curve Theorem is non-vacuously satisfied is a cyclicly connected subset of a plane or a sphere.*[§]

THEOREM 84. *A locally connected, locally peripherally compact, complete metric space in which the Jordan Curve Theorem is non-vacuously satisfied is a cyclicly connected subset of a plane or a sphere.*

[†] J. H. Roberts, *Concerning compact continua in certain spaces of R. L. Moore*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 615–621

[‡] Axiom 4.

[§] See p 53.

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NON-SEPARABLE AND PLANAR GRAPHS*

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Introduction. In this paper the structure of graphs is studied by purely combinatorial methods. The concepts of rank and nullity are fundamental. The first part is devoted to a general study of non-separable graphs. Conditions that a graph be non-separable are given; the decomposition of a separable graph into its non-separable parts is studied; by means of theorems on circuits of graphs, a method for the construction of non-separable graphs is found, which is useful in proving theorems on such graphs by mathematical induction. In the second part, a dual of a graph is defined by combinatorial means, and the paper ends with the theorem that a necessary and sufficient condition that a graph be planar is that it have a dual.

The results of this paper are fundamental in papers by the author on *Congruent graphs and the connectivity of graphs*† and on *The coloring of graphs*.‡

I. NON-SEPARABLE GRAPHS

1. **Definitions.**§ A *graph G* consists of two sets of symbols, finite in number: *vertices*, $a, b, c, \cdots, f$, and *arcs*, $\alpha(ab), \beta(ac), \cdots, \delta(cf)$. If an arc $\alpha(ab)$ is present in a graph, its *end vertices* a, b are also present. We may write an arc $\alpha(ab)$ or $\alpha(ba)$ at will; we may write it also ab or ba if no confusion arises,—if there is but a single arc joining a and b in G. We say the vertices a and b are *on* the arc $\alpha(ab)$, and the arc $\alpha(ab)$ is *on* the vertices a and b. The *null graph* is the graph containing no arcs or vertices.

The obvious geometrical interpretation of such a graph, or *abstract graph*, is a *topological graph*, let us say. Corresponding to each vertex of the abstract graph, we select a point in 3-space, a vertex of the topological graph. Corresponding to each arc $\alpha(ab)$ of the abstract graph, we select an arc joining the corresponding vertices of the topological graph. An arc is here a set of points in (1, 1) correspondence with the unit interval, its end vertices corresponding

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* Presented to the Society, October 25, 1930; received by the editors February 2, 1931. An outline of this paper will be found in the Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 125–127.

† American Journal of Mathematics, vol. 54 (1932), pp. 150–168.

‡ An outline will be found in the Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 122–125.

§ Compare Ste. Laguë, *Les Réseaux*, Mémorial des Sciences Mathématiques, fascicule 18, Paris, 1926.

to be suspended in G if C includes
two or more arcs, and all vertices
belonging to at least two arcs of
not in C and no other vertices belong
to any arc of G not in C,

with the ends of the interval. Moreover, we let no arc pass through other vertices or intersect other arcs. We shall consider topological graphs no further till we come to the section on planar graphs.

An *isolated vertex* is a vertex which is not on any arc. A *chain* is a set of one or more distinct arcs which can be ordered thus: ab, bc, cd, $\cdots$, ef, where vertices in different positions are distinct, i.e. the chain may not intersect itself. A *suspended chain* is a chain containing two or more arcs such that no vertex of the chain other than the first and last is on other arcs, and these two vertices are each on at least two other arcs. A *circuit* is a set of one or more distinct arcs which can be put in cyclic order, ab, bc, $\cdots$, ef, fa, vertices being distinct as in the case of the chain. A *k-circuit* is a circuit containing k arcs. Thus, the arc $\alpha(aa)$ is a 1-circuit; the two arcs $\alpha(ab)$, $\beta(ab)$ form a 2-circuit.

A graph is *connected* if any two of its vertices are joined by a chain. Obviously, if a and b are joined by a chain, and b and c are joined by a chain, then a and c are joined by a chain. Any graph consists of a certain number of *connected pieces* (one, if the graph is connected). In particular, an isolated vertex is one of the connected pieces of a graph. A graph is called *cyclicly connected* if any two of its vertices are contained in a circuit. If G_1, G_2, $\cdots$, G_m are a set of graphs, no two of which have a common vertex (or arc, therefore), we say the graph G, formed of the arcs and vertices of all these graphs, is the *sum* of these graphs. Thus, a graph is the sum of its connected pieces. A *forest* is a graph containing no circuit. A *tree* is a connected forest. A *subgraph H* of G is a graph containing a subset (in particular, all or none), of the arcs of G, and those vertices of G which are on these arcs.

2. **Rank and nullity.**[*] Given a graph G which contains V vertices, E arcs, and P connected pieces, we define its *rank R*, and its *nullity* (or *cyclomatic number* or first Betti number) N, by the equations

$$R = V - P,$$

$$N = E - R = E - V + P.$$

If G contains the single arc ab, it is of rank 1, nullity 0, while if it contains the single arc aa, it is of rank 0, nullity 1.

The first two theorems follow immediately from the definitions of rank and nullity:

THEOREM 1. *If isolated vertices be added to or subtracted from a graph, the rank and nullity remain unchanged.*

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[*] These are just the rank and nullity of the matrix H_1 of Poincaré. See Veblen's Colloquium Lectures, *Analysis Situs.*

THEOREM 2. *Let the graph G' be formed from the graph G by adding the arc ab. Then*

(1) *if a and b are in the same connected piece in G, then*

$$R' = R, \quad N' = N + 1;$$

(2) *if a and b are in different connected pieces in G, then*

$$R' = R + 1, \quad N' = N.$$

THEOREM 3. *In any graph G,*

$$R \geqq 0, \quad N \geqq 0.$$

For let G_1 be the graph containing the vertices of G but no arcs. Then if R_1 and N_1 are its rank and nullity,

$$R_1 = N_1 = 0.$$

We build up G from G_1 by adding the arcs one at a time. The theorem now follows from Theorem 2.

THEOREM 4. *A forest G is a graph of nullity* 0, *and conversely.*

Suppose first G contained a circuit P. We shall show that the nullity of G is >0. We build up G arc by arc, adding first the arcs of the circuit P. In adding the last arc of the circuit, the nullity is increased by 1, as this arc joins two vertices already connected. (This argument holds even if the circuit is a 1-circuit.) But in adding the rest of the arcs, the nullity is never decreased, by Theorem 2. Thus the nullity of G is >0.

Now suppose G is a forest, and therefore contains no circuit. Build up G arc by arc. Each arc we add joins two vertices formerly not connected. For otherwise, this arc, together with the arcs of a chain connecting the two vertices, would form a circuit. Therefore, by Theorem 2, the nullity remains always the same, and is thus 0.

3. **Theorems on non-separable graphs.** We introduce the following

Definitions. Let H_1, which contains the vertex a_1, and H_2, which contains the vertex a_2, be two graphs without common vertices. Let us rename a_1 a, and rename the arcs of H_1 on a_1 accordingly; that is, if a_1b is an arc on a_1, we rename it ab. Rename also a_2 a, and rename the arcs of H_2 accordingly. H_1 and H_2 have now the vertex a in common; they form the graph G, say. We say G is formed by letting the vertex a_1 of H_1 *coalesce* with the vertex a_2 of H_2, or, by joining H_1 and H_2 at a vertex. Geometrically, we pull the vertices a_1 and a_2 together to form the single vertex a.

Let G be a connected graph such that there exist no two graphs H_1 and

H_2, each containing at least one arc, which form G if they are joined at a vertex. Then G is called *non-separable*. Geometrically, a connected graph is non-separable if we cannot break it at a single vertex into two graphs, each containing an arc. For example, the graph consisting of the two arcs ab, bc is separable, as is the graph consisting of the two arcs $\alpha(aa)$, $\beta(aa)$. A graph containing but a single arc is non-separable, as is the graph containing only the arcs $\alpha(ab)$, $\beta(ab)$.

If G is not non-separable, we say G is *separable*. Thus, a graph that is not connected is separable. Suppose some connected piece G_1 of G is separable. If H_1 and H_2 joined at the vertex a form G_1, we say a is a *cut vertex* of G. We have consequently

THEOREM 5. *A necessary and sufficient condition that a connected graph be non-separable is that it have no cut vertex.*

THEOREM 6. *Let G be a connected graph containing no 1-circuit. A necessary and sufficient condition that the vertex a be a cut vertex of G is that there exist two vertices b, c in G, each distinct from a, such that every chain from b to c passes through a.*

First suppose a is a cut vertex of G. Then, by definition, H_1 and H_2, each containing at least one arc which is not a 1-circuit, form G if they are joined at a. Let b be a vertex of H_1 and c a vertex of H_2, each distinct from a. As a is the only vertex in both H_1 and H_2, every chain from b to c in G passes through a.

Suppose now every chain from b to c in G passes through a. Remove the vertex a and all the arcs on a. The resulting graph G' is not connected, b and c being in different connected pieces. Let H_1' be that connected piece of G' containing b, and let H_2' be the rest of G'. Replace a by the two vertices a_1 and a_2. Now put back the arcs we removed, letting them touch a_1 if their other end vertices are in H_1', and letting them touch a_2 otherwise. Let H_1 and H_2 be the resulting graphs. Then H_1 and H_2 each contain at least one arc, and they form G if the two vertices a_1, a_2 are made to coalesce. Hence, by definition, a is a cut vertex of G.

THEOREM 7. *Let G be a graph containing no 1-circuit and containing at least two arcs. A necessary and sufficient condition that G be non-separable is that it be cyclicly connected.*[*]

If G is not connected, the theorem is obvious. Assume therefore G is connected.

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[*] A similar theorem has been proved for more general continuous curves by G. T. Whyburn, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 429–433.

Suppose first G is separable. Then, by Theorem 5, G has a cut vertex a, and by Theorem 6, there are two vertices b, c in G such that every chain from b to c passes through a. Hence there is no circuit in G containing b and c.

Suppose now there exist two vertices b, c in G which are contained in no circuit. Let $bd, de, \cdots, gc$ be some chain from b to c.

Case 1. There exists a circuit containing b and d. In this case, let a be the last vertex of the chain which is contained in a circuit passing also through b. Let f be the next vertex of the chain. Then every chain from f to b passes through a. For suppose the contrary. Let C be a chain from f to b not passing through a. Let P be a circuit containing b and a. Follow C from f till we first reach a vertex of P. Follow the circuit P now as far as b if b was not the vertex we reached, and continue along P till we reach a. Passing from a to f along the arc af completes a circuit containing both b and f, contrary to hypothesis. Hence, by Theorem 6, a is a cut vertex of G, and therefore G is separable.

Case 2. There exists no circuit containing b and d. Then there is but a single arc joining b and d, and they are joined by no other chain. As G is connected and contains at least two arcs, there is either another arc on b or another arc on d, say the first. The other case is exactly similar. If we add a vertex b' and replace the arc bd by the arc $b'd$, b and d are no longer joined by a chain, and hence the resulting graph G' is not connected. Let H_1 be that part of G' containing the arc $b'd$, and let H_2 be the rest of G'. As there is still an arc on b, H_2 contains at least one arc. Letting the vertices b and b' coalesce forms G, and hence G is separable. The proof is now complete.

THEOREM 8. *A non-separable graph G containing at least two arcs contains no 1-circuit and is of nullity >0. Each vertex is on at least two arcs.*

Suppose G contained a 1-circuit. Call it H_1. Let H_2 be the rest of the graph. Then H_1 and H_2 have but a single vertex in common, and thus G is separable.

Next, by Theorem 7, G is cyclicly connected. As G contains no 1-circuit, G contains at least two vertices. Containing these there is a circuit. Therefore, by Theorem 4, the nullity of G is >0.

Finally, if there were a vertex on no arcs, G would not be connected. If there were a vertex a on the single arc ab, b would be a cut vertex of G.

THEOREM 9. *Let G be a graph of nullity 1 containing no isolated vertices, such that the removal of any arc reduces the nullity to 0. Then G is a circuit.*

By Theorem 4, G contains a circuit. Suppose G contained other arcs besides. Removing one of these, the nullity remains 1, as the circuit is still present, contrary to hypothesis. There are no other vertices in G, as G contains no isolated vertices. Hence G is just this circuit.

THEOREM 10. *A non-separable graph G of nullity* 1 *is a circuit.*

If G contains but a single arc, it is a 1-circuit, being of nullity 1. Suppose G contains at least two arcs. By Theorem 8, it contains no 1-circuit. By Theorem 7, it is cyclicly connected. Remove any arc ab from G; a and b are still connected, and therefore, by Theorem 2, the nullity of G is reduced to 0. Hence, by Theorem 9, G is a circuit.

The converses of the last two theorems are obviously true.

4. Decomposition of separable graphs. If the graph G contains a connected piece which is separable, we may separate that piece into two graphs, these graphs having formerly but a single vertex in common. We may continue in this manner until every resulting piece of G is non-separable. We say G is separated into its *components.*

LEMMA. *Let the connected separable graph G be decomposed into the two pieces H_1 and H_2 which had only the vertex a in common in G. Then every non-separable subgraph of G is contained wholly in either H_1 or H_2.*

Suppose the contrary. Then some non-separable subgraph I of G is not contained wholly in either H_1 or H_2. Let I_1 be that part of I in H_1, and I_2 that part in H_2; I_1 and I_2 have at most the vertex a in common. I_1 and I_2 each contain at least one arc. For otherwise, if I_1, say, contained no arc, as it contains a vertex distinct from a, it would not be connected. Thus I is separable into the pieces I_1 and I_2, a contradiction again.

THEOREM 11. *Every non-separable subgraph of G is contained wholly in one of the components of G.*

This follows upon repeated application of the above lemma.

THEOREM 12. *A graph G may be decomposed into its components in a unique manner.*

Suppose we could decompose G into the components $H_1, H_2, \cdots, H_m$, and also into the components $H_1', H_2', \cdots, H_n'$. We shall show that these sets are identical. Take any H_i. It is a non-separable subgraph of G, and thus is contained in some component H_j', by Theorem 11. Similarly, H_j' is contained in some component H_k. Thus H_i is contained in H_k, and they are therefore identical. Hence H_i and H_j' are identical. In this manner we show that each H_k is identical with some H_l', and each H_l' is identical with some H_k, proving the theorem.

THEOREM 13. *Let $H_1, H_2, \cdots, H_m$ be the components of G. Let $R_1, R_2, \cdots, R_m$, and $N_1, N_2, \cdots, N_m$ be their ranks and nullities. Then*

$$R = R_1 + R_2 + \cdots + R_m,$$

$$N = N_1 + N_2 + \cdots + N_m.$$

Let G' be G separated into its components, and let R' be the rank of G'. G is formed from G' by letting vertices of different components coalesce. Each time we join two pieces, the number of vertices and the number of connected pieces are each reduced by 1, so that the rank remains the same. Thus

$$R = R'.$$

Now

$$V' = V_1 + V_2 + \cdots + V_m,$$

$$P' = P_1 + P_2 + \cdots + P_m$$

(where each $P_i = 1$). Subtracting,

$$R = R' = R_1 + R_2 + \cdots + R_m.$$

As also

$$E = E_1 + E_2 + \cdots + E_m,$$

it follows that

$$N = N_1 + N_2 + \cdots + N_m.$$

For a converse of this theorem, see Theorem 17.

THEOREM 14. *Divide the arcs of the non-separable graph G into two groups, each containing at least one arc, forming the subgraphs H_1 and H_2, of ranks R_1 and R_2. Then*

$$R_1 + R_2 > R.$$

Let the connected pieces of H_1 be $H_{11}, \cdots, H_{1m}$ (there may be but one piece, H_{11}), and let those of H_2 be $H_{21}, \cdots, H_{2n}$. Then obviously

$$R_1 = R_{11} + \cdots + R_{1m},$$

$$R_2 = R_{21} + \cdots + R_{2n},$$

whence

$$R_1 + R_2 = R_{11} + \cdots + R_{1m} + R_{21} + \cdots + R_{2n}.$$

Let G' be the sum of the graphs $H_{11}, \cdots, H_{2n}$. Then G' is of rank $R_{11} + \cdots + R_{2n}$. We form G from G' by letting vertices of the graphs $H_{11}, \cdots, H_{2n}$ coalesce. Each time we let vertices of different connected pieces coalesce, the rank is unaltered. Each time we let vertices in the same connected piece coalesce, the rank is reduced by 1. This latter operation happens at least once. For otherwise, let a_1 and a_2 be the last two vertices we let coalesce. Then a_1 and a_2 were formerly in two different pieces, I_1 and I_2. Thus

I_1 and I_2 joined at a vertex form G, and G is separable, contrary to hypothesis. Thus the rank of G is less than the rank of G', that is,

$$R < R_{11} + \cdots + R_{2n}.$$

Hence

$$R_1 + R_2 > R.^*$$

Theorems 13 and 14 give

THEOREM 15. *A necessary and sufficient condition that a graph be non-separable is that there exist no division of its arcs into two groups H_1 and H_2, each containing at least one arc, so that*

$$R = R_1 + R_2.$$

5. **Circuits of graphs.** We shall say two non-separable graphs, each containing at least one arc, form a *circuit of graphs*, if they have at least two common vertices. (They may also have common arcs.) Thus the two graphs G_1: $\alpha(ab)$ and G_2: $\alpha(ab)$ (which are the same graph) form a circuit of graphs. However, the two graphs G_1: $\alpha(aa)$ and G_2: $\beta(aa)$, having but one common vertex, do not form a circuit of graphs. We shall say three or more non-separable graphs form a *circuit of graphs* if we can name them $G_1, G_2, \cdots, G_m$ in such a way that G_1 and G_2 have just the vertex a_1 in common, G_2 and G_3 have just the vertex a_2 in common, $\cdots$, G_m and G_1 have just the vertex a_m in common, these vertices are all distinct, and no other two of these graphs have a common vertex. Thus the three graphs G_1:ab, G_2:bc, G_3:ca form a circuit of graphs.

We note that there can be no 1-circuit in a circuit of graphs; also, no subset of the graphs in a circuit of graphs form a circuit of graphs. We may think of a circuit of graphs as forming a single graph.

THEOREM 16. *A circuit of graphs G is a non-separable graph.*

First suppose there are but two graphs, G_1 and G_2, present. Suppose G were separable. Then it is separable into at least two components $H_1, H_2, \cdots, H_k$. By Theorem 11, G_1 and G_2 are each contained wholly in one of these components. As G_1 and G_2 together form G, there are just two components, and they are G_1 and G_2. These, when joined at a vertex, form G. But this is contrary to the hypothesis that G_1 and G_2 have at least two vertices in common.

Next suppose there are more than two graphs present. Let C_1 be a chain in G_1 joining a_m and a_1, let C_2 be a chain in G_2 joining a_1 and a_2, $\cdots$, let C_m be a chain in G_m joining a_{m-1} and a_m. These chains taken together form a circuit P passing through all the graphs. Now separate G into its components. By Theorem 11 (see the converse of Theorem 10), P is contained in one of these components. The same is true of each of the graphs $G_1, G_2, \cdots, G_m$, and hence these graphs are all contained in the same component. Thus G is itself this component, that is, G is non-separable.

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* This theorem may also be proved easily from Theorem 17.

THEOREM 17. *Let* $G_1, \cdots, G_m$ *be a set of non-separable graphs, each containing at least one arc, and let G be formed by letting vertices and arcs of different graphs coalesce. Then the following four statements are all equivalent*:

(1) $G_1, \cdots, G_m$ *are the components of* G.

(2) *No two of the graphs* $G_1, \cdots, G_m$ *have an arc in common, and there is no circuit in G containing arcs of more than one of these graphs.*

(3) *No subset of these graphs form a circuit of graphs.*

(4) *If* $R, R_1, \cdots, R_m$ *are the ranks of* $G, G_1, \cdots, G_m$ *respectively, then*

$$R = R_1 + \cdots + R_m.$$

We note that we cannot replace the word rank by the word nullity in (4). For let G be the graph containing the arcs $\alpha(ab)$, $\beta(ab)$, $\gamma(ab)$. Let G_1 contain α and β, and G_2, β and γ. Then the nullity of G is the sum of the nullities of G_1 and G_2, but G_1 and G_2 are not the components of G. We shall prove

(a) if (1) holds, (2) holds,

(b) if (2) holds, (3) holds,

(c) if (3) holds, (1) holds, establishing the equivalence of (1), (2) and (3);

(d) if (1) holds, (4) holds, and finally

(e) if (4) holds, (3) holds, establishing the equivalence of (4) and the other statements.

(a) If (1) holds, (2) holds. For first, in forming G from its components $G_1, \cdots, G_m$, we let vertices alone coalesce, and thus no two of the graphs have an arc in common. Also, there is no circuit in G containing arcs of more than one of the graphs; for each circuit, being a non-separable graph, is contained entirely in one of the components of G, by Theorem 11.

(b) If (2) holds, (3) holds. For suppose the contrary. If, first, some two graphs, say G_1 and G_2, form a circuit of graphs, they have at least two vertices in common, say a and b. Join a and b by a chain C in G_1 and by a chain D in G_2. By hypothesis, G_1 and G_2 have no arcs in common, and thus the arcs of C and D are distinct. From a follow along C till we first reach a vertex d of D. From d follow along D till we get back to a. We have formed thus a circuit containing arcs of both G_1 and G_2, contrary to hypothesis.

Now suppose the graphs $G_1, \cdots, G_k$, $k > 2$, formed a circuit of graphs. In the proof of Theorem 16 we found a circuit passing through all the graphs of such a circuit of graphs, again contrary to hypothesis.

(c) If (3) holds, (1) holds. Assuming that no subset of the graphs $G_1, \cdots, G_m$ forms a circuit of graphs, we will show first that some one of these graphs has at most a single vertex in common with other of the graphs. For suppose each graph had at least two vertices in common with other graphs. Then G_1 has a vertex a_1 in common with some graph, say G_2. As G_2 has at least two vertices in common with other graphs, it has a vertex a_2, distinct from a_1, in common with another graph, say G_3. If we continue in this manner, we must at some point get back to a graph we have already considered.

Now starting with G_1, consider the graphs in order, and let G_i be the first one which has a vertex in common with one of the preceding graphs other than the vertex a_{i-1}, which we know already it has in common with G_{i-1}. Now of the graphs $G_{i-1}, G_{i-2}, \cdots, G_1$, let G_j be the first with which G_i has a common vertex, other than the vertex a_{i-1}. First suppose G_j is G_{i-1}. Then G_i and G_{i-1} have at least two vertices in common, and they form therefore a circuit of graphs, contrary to hypothesis. Next suppose G_j is not G_{i-1}.Then on account of the choice of G_i and G_j, G_j and G_{j+1} have just one common vertex a_j, G_{j+1} and G_{j+2} have just one common vertex $a_{j+1}, \cdots$, G_i and G_j have just one common vertex a_i (for otherwise G_i and G_j would form a circuit of graphs), and no other two of these graphs have a vertex in common. These vertices $a_j, a_{j+1}, \cdots, a_i$ are all distinct. For, on account of the construction of the chain of graphs, two succeeding vertices a_k and a_{k+1} are distinct. a_i and a_j are distinct, for otherwise G_i and G_{j+1} would have a common vertex, etc. These graphs $G_j, G_{j+1}, \cdots, G_i$ form therefore a circuit of graphs, contrary to hypothesis.

Some graph therefore, say G_1, has at most a single vertex in common with the other graphs. Thus either it is separated from them, or we can separate it at a single vertex. Now among the graphs $G_2, \cdots, G_m$, there is also no circuit of graphs, so again we can separate one of them, say G_2. Continuing, we have finally separated G into its components $G_1, G_2, \cdots, G_m$.

(d) If (1) holds, (4) holds. This is just Theorem 13.

(e) If (4) holds, (3) holds. Let G' be the sum of the graphs $G_1, \cdots, G_m$. We form G from G' by letting vertices and arcs of different graphs coalesce. Each time we let two vertices coalesce, either (α) the two vertices were formerly in different connected pieces, in which case the rank is unchanged, or (β) the two vertices were in the same connected piece, in which case the rank is reduced by 1. Letting arcs alone coalesce (their end vertices having already coalesced) does not alter the rank. Thus in any case, the rank is never increased. To begin with, the rank of G' is $G_1+ \cdots +G_m$, and by hypothesis, the rank of G is $G_1+ \cdots +G_m$. Thus the rank is never altered, and (β) never occurs. Hence, obviously, no circuit of graphs is formed in forming G from G'. This completes the proof of the theorem.

6. **Construction of non-separable graphs.** We prove the following theorem:

THEOREM 18. *If G is a non-separable graph of nullity $N>1$, we can remove an arc or suspended chain from G, leaving a non-separable graph G' of nullity $N-1$.*

Assume the theorem is true for all graphs of nullity $2, 3, \cdots, N-1$. We shall prove it for any graph of nullity N (including the case where $N=2$). This will establish the theorem in general.

Take any non-separable graph G of nullity $N>1$. It contains at least two arcs, and therefore, by Theorem 8, it contains no 1-circuit. Remove from G any arc ab, forming the graph G_1. If G_1 is non-separable, we are through. Suppose therefore G_1 is separable, and let its components be $H_1, H_2, \cdots, H_{m-1}$. G_1 is connected, for between any two vertices c, d there exists a circuit in G by Theorem 7, and therefore there is a chain joining them in G_1.

Let H_m consist of the arc ab. By Theorem 17, no subset of the graphs $H_1, \cdots, H_{m-1}$ form a circuit of graphs, while some subset of the graphs $H_1, \cdots, H_m$ form a circuit of graphs. We shall show that the whole set of graphs $H_1, \cdots, H_m$ form a circuit of graphs. Otherwise, some proper subset, which includes H_m, form a circuit of graphs.

Let H be the graph formed from this circuit of graphs by dropping out H_m. By Theorem 16, the circuit of graphs is a non-separable graph; hence H is connected. All the arcs in G_1 not in the circuit of graphs, form a graph I. Let I_1 be a connected piece of I. Then I_1 has at most a single vertex in common with the rest of G. For suppose I_1 had the two vertices c and d in common with H. From c follow along some chain towards d in H till we first reach a vertex e in I_1. From e follow back along some chain in I_1 to c. We have formed thus a circuit containing arcs of both H and I_1. But as H consists of a certain subset of the components of G_1, this circuit contains arcs of at least two components of G_1, contrary to Theorem 17. Thus I_1 has at most a single vertex in common with the rest of G, and hence G is separable, contrary to hypothesis. Thus $H_1, \cdots, H_m$ form a circuit of graphs, that is, G is formed of a circuit of graphs.

As we assumed G_1 was separable, $m \geqq 3$. Therefore we can order the graphs so that H_1 and H_2 have just the vertex a_1 in common, $\cdots$, H_{m-1} and H_m have just the vertex $a_{m-1}=b$ in common, and H_m and H_1 have just the vertex $a_m=a$ in common. Moreover, these vertices are all distinct, and no other two of the graphs $H_1, \cdots, H_m$ have a common vertex.

As the nullity of G was >1, the nullity of G_1 is >0. By Theorem 13, this is the sum of the nullities of $H_1, \cdots, H_{m-1}$. Therefore the nullity of some one of these graphs, say H_i, is >0.

Suppose first the nullity of H_i is 1. Then, by Theorem 10, H_i is a circuit, consisting of two chains joining a_{i-1} and a_i. Remove one of these chains from G. This leaves a graph G', which again is a circuit of graphs. For the graph H_i we replace by an ordered set of non-separable graphs, each consisting of one of the arcs of the chain we have left in H_i.

Suppose next the nullity of H_i is >1. It is less than N, as H_i is contained in G_1, whose nullity is $N-1$. Therefore, by induction, we can remove an arc or a suspended chain, leaving a non-separable graph H_i' of nullity one less. If neither a_{i-1} nor a_i has thus been removed, we again have a circuit of graphs. Suppose a_i but not a_{i-1} was removed. Replace that part of the chain we removed joining a_i and a vertex of H_i distinct from a_{i-1}. Here again we have a circuit of graphs, H_i being replaced by H_i' and a set of arcs. The case is the same if a_{i-1} but not a_i was removed. If finally, both a_i and a_{i-1} were in the chain we removed, we put back all of the chain but that part between these two vertices. Here again, the resulting graph G' is a circuit of graphs.

Thus in all cases we can drop out from G an arc or suspended chain, leaving a circuit of graphs. By Theorem 16, the resulting graph G' is non-separable. As also the nullity of G' is one less than the nullity of G, the theorem is now proved.

As a consequence of this theorem, Theorem 8, and Theorem 10, we have

THEOREM 19. *We can build up any non-separable graph containing at at least two arcs by taking first a circuit, then adding successively arcs or suspended chains, so that at any stage of the construction we have a non-separable graph.*

It is easily seen that, conversely, any graph built up in this manner is non-separable. For each time we add an arc or suspended chain, these arcs, each considered as a graph, together with the non-separable graph already present, form a circuit of graphs.

II. DUALS, PLANAR GRAPHS

7. Congruent graphs. We introduce the following

Definitions. Given two graphs G and G', if we can rename the vertices and arcs of one, giving distinct vertices and distinct arcs different names, so that it becomes identical with the other, we say the two graphs are *congruent*.* (We used formerly the word "homeomorphic.")

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* See the author's American Journal paper, cited in the introduction.

The geometrical interpretation is that we can bring the two graphs into complete coincidence by a (1, 1) continuous transformation.

Two graphs are called *equivalent* if, upon being decomposed into their components, they become congruent, except possibly for isolated vertices.

8. **Duals.** Given a graph G, if H_1 is a subgraph of G, and H_2 is that subgraph of G containing those arcs not in H_1, we say H_2 is the *complement* of H_1 in G.

Throughout this section, R, R', r, r', etc., will stand for the ranks of G, G', H, H', etc., respectively, with similar definitions for V, E, P, N.

Definition. Suppose there is a (1, 1) correspondence between the arcs of the graphs G and G', such that if H is any subgraph of G and H' is the complement of the corresponding subgraph of G', then

$$r' = R' - n.$$

We say then that G' is a *dual* of G.*

Thus, if the nullity of H is n, then H' (including all the vertices of G') is in n more connected pieces than G'.

THEOREM 20. *Let G' be a dual of G. Then*

$$R' = N,$$
$$N' = R.$$

For let H be that subgraph of G consisting of G itself. Then

$$n = N.$$

If H' is the complement of the corresponding subgraph of G', H' contains no arcs, and is the null graph. Thus

$$r' = 0.$$

But as G' is a dual of G,

$$r' = R' - n.$$

These equations give

$$R' = N.$$

The other equation follows when we note that $E' = E$.

THEOREM 21. *If G' is a dual of G, then G is a dual of G'.*

Let H' be any subgraph of G', and let H be the complement of the corresponding subgraph of G. Then, as G' is a dual of G,

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* While this definition agrees with the ordinary one for graphs lying on a plane or sphere, a graph on a surface of higher connectivity, such as the torus, has in general no dual. (See Theorems 29 and 30.)

$$r' = R' - n.$$

By Theorem 20,

$$R' = N.$$

We note also,

$$e + e' = E.$$

These equations give

$$\begin{aligned} r = e - n &= e - (R' - r') = e - N + (e' - n') \\ &= E - N - n' = R - n'. \end{aligned}$$

Thus G is a dual of G'.

Whenever we have shown that one graph is a dual of another graph, we may now call the graphs "dual graphs."

Lemma. *If a graph G is decomposed into its components, the rank and nullity of any subgraph H is left unchanged.*

For each time we separate G at a vertex, H is either unchanged or is separated at a vertex. Hence neither its rank nor its nullity is altered. (See the proof of Theorem 13.)

Theorem 22. *If G' and G'' are equivalent and G' is a dual of G, then G'' is a dual of G.*

Let H be any subgraph of G, and let H' be the complement of the corresponding subgraph of G'. Let G_1' and G_1'' be G' and G'' decomposed into their components. Then G_1' and G_1'' are congruent. H' turns into a subgraph H_1' of G'. Let H_1'' be the corresponding subgraph of G_1'', and H'' the same subgraph in G''. Then

$$r_1' = r_1''.$$

But by the above lemma,

$$r' = r_1', \qquad r'' = r_1''.$$

Hence

$$r' = r''.$$

As a special case of this equation, letting H' be the whole of G', we have

$$R' = R''.$$

As G' is a dual of G,

$$r' = R' - n.$$

Therefore

$$r'' = R'' - n,$$

and G'' is a dual of G.

The converse of this theorem is not true. For define the three graphs

$G: \alpha(ab), \beta(ab), \gamma(ac), \delta(cb), \epsilon(ad), \zeta(db);$

G': $\alpha'(a'b'), \beta'(c'd'), \gamma'(a'd'), \delta'(a'd'), \epsilon'(b'c'), \zeta'(b'c')$;
G'': $\alpha''(a''b''), \beta''(b''c''), \gamma''(a''d''), \delta''(a''d''), \epsilon''(c''d''), \zeta''(c''d'')$.
G' and G'' are both duals of G, but they are not congruent.*

THEOREM 23. *Let $G_1, \cdots, G_m$ and $G_1', \cdots, G_m'$ be the components of G and G' respectively, and let G_i' be a dual of G_i, $i=1, \cdots, m$. Then G' is a dual of G.*

Let H be any subgraph of G, and let the parts of H in $G_1, \cdots, G_m$ be $H_1, \cdots, H_m$. Let H_i' be the complement of the subgraph corresponding to H_i in G_i', $i=1, \cdots, m$, and let H' be the union of $H_1', \cdots, H_m'$ in G'. Then H' is the complement of the subgraph in G' corresponding to H in G. Using the proof of Theorem 13, we find that

$$r' = r_1' + \cdots + r_m',$$

and

$$n = n_1 + \cdots + n_m.$$

As also

$$R' = R_1' + \cdots + R_m'$$

and

$$r_i' = R_i' - n_i \qquad (i = 1, \cdots, m),$$

adding these last equations gives

$$r' = R' - n,$$

and hence G' is a dual of G.

THEOREM 24. *Let $G_1, \cdots, G_m$ and $G_1', \cdots, G_m'$ be the components of the dual graphs G and G', and let the correspondence between these two graphs be such that arcs in G_i correspond to arcs in G_i', $i=1, \cdots, m$. Then G_i and G_i' are duals, $i=1, \cdots, m$.*

Let H_1 be any subgraph of G_1, let H' be the complement of the corresponding subgraph in G', and let H_1' be the complement in G'. Then $H_1', G_2', \cdots, G_m'$ form H'. By Theorem 13, we find

$$R' = R_1' + R_2' + \cdots + R_m'$$

and

$$r' = r_1' + R_2' + \cdots + R_m'.$$

Now

$$r' = R' - n_1,$$

hence

$$r_1' = R_1' - n_1,$$

and G_1' is a dual of G_1. Similarly for $G_2', \cdots, G_m'$.

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* See the author's American Journal paper, however.

THEOREM 25. *Let G and G' be dual graphs, and let $H_1, \cdots, H_m$ be the components of G. Let $H_1', \cdots, H_m'$ be the corresponding subgraphs of G'. Then $H_1', \cdots, H_m'$ are the components of G', and H_i' is a dual of H_i, $i=1, \cdots, m$.*

H_1 is the subgraph of G corresponding to H_1' in G'. Its complement is I_1, the graph formed of the arcs of $H_2, \cdots, H_m$. Obviously $H_2, \cdots, H_m$ are the components of I_1. Hence, by Theorem 13, the nullity of I_1 is $n_2+n_3+\cdots+n_m$. Thus, as G' is a dual of G,

$$r_1' = R' - (n_2 + n_3 + \cdots + n_m).*$$

Similarly,

$$r_2' = R' - (n_1 + n_3 + \cdots + n_m),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$r_m' = R' - (n_1 + n_2 + \cdots + n_{m-1}).$$

Adding these equations gives

$$r_1' + r_2' + \cdots + r_m' = mR' - (m - 1)(n_1 + n_2 + \cdots + n_m).$$

As $H_1, H_2, \cdots, H_m$ are the components of G,

$$N = n_1 + n_2 + \cdots + n_m.$$

Also, as G and G' are duals, by Theorem 20,

$$R' = N.$$

Hence

$$\begin{aligned} r_1' + r_2' + \cdots + r_m' &= mR' - (m - 1)R' \\ &= R'. \end{aligned}$$

Let now $H_{11}', \cdots, H_{1k_1}'$ be the components of H_1' (there may be but one) and similarly for $H_2', \cdots, H_m'$. Then, by Theorem 13,

$$r_1' = r_{11}' + \cdots + r_{1k_1}',$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$r_m' = r_{m1}' + \cdots + r_{mk_m}'.$$

Adding these equations gives

$$\sum_{i,j} r_{ij}' = r_1' + \cdots + r_m' = R'.$$

As the graphs $H_{11}', \cdots, H_{mk_m}'$ are non-separable, Theorem 17 tells us that they are the components of G'. Hence G' has at least as many components as

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* Which equals n_1.

G. Similarly, G has at least as many components as G'. They have therefore the same number, m, of components.

There are therefore m graphs in the set $H_{11}', \cdots, H_{mk_m}'$. But there is at least one such graph in each graph $H_1', \cdots, H_m'$, and there is therefore exactly one in each. Hence each graph H_{i1}' fills out the graph H_i', and the two sets of graphs $H_{11}', \cdots, H_{mk_m}'$ and $H_1', \cdots, H_m'$ are identical, that is, $H_1', \cdots, H_m'$ are the components of G'.

The rest of the theorem follows from Theorem 24.

As a special case of this theorem, we have

THEOREM 26. *A dual of a non-separable graph is non-separable.*

9. **Planar graphs.** Up till now, we have been considering abstract graphs alone. However, the definition of a planar graph is topological in character. This section may be considered as an application of the theory of abstract graphs to the theory of topological graphs.

Definitions. A topological graph is called *planar* if it can be mapped in a (1, 1) continuous manner on a sphere (or a plane). For the present, we shall say that an abstract graph is *planar* if the corresponding topological graph is planar. Having proved Theorem 29, we shall be justified in using the following purely combinatorial definition: *A graph is planar if it has a dual.*

We shall henceforth talk about "graphs" simply, the terms applying equally well to either abstract or topological graphs.

LEMMA. *If a graph can be mapped on a sphere, it can be mapped on a plane, and conversely.*

Suppose we have a graph mapped on a sphere. We let the sphere lie on the plane, and rotate it so that the new north pole is not a point of the graph. By stereographic projection from this pole, the graph is mapped on the plane. The inverse of this projection maps any graph on the plane onto the sphere.

By the *regions* of a graph lying on a sphere or in a plane is meant the regions into which the sphere or plane is thereby divided. A given region of the graph is characterized by those arcs of the graph which form its boundary. If the graph is in a plane, the outside region is the unbounded region.

LEMMA. *A planar graph may be mapped on a plane so that any desired region is the outside region.*

We map the graph on a sphere, and rotate it so that the north pole lies inside the given region. By stereographic projection, the graph is mapped onto the plane so that the given region is the outside region.

We return now to the work in hand.

THEOREM 27. *If the components of a graph G are planar, G is planar.*

Suppose the graphs G_1 and G_2 are planar, and G' is formed by letting the vertices a_1 and a_2 of G_1 and G_2 coalesce. We shall show that G' is planar. Map G_1 on a sphere, and map G_2 on a plane so that one of the regions adjacent to the vertex a_2 is the outside region. Shrink the portion of the plane containing G_2 so it will fit into one of the regions of G_1 adjacent to a_1. Drawing a_1 and a_2 together, we have mapped G' on the sphere.* The theorem follows as a repeated application of this process.

THEOREM 28. *Let G and G' be dual graphs, and let $\alpha(ab)$, $\alpha'(a'b')$ be two corresponding arcs. Form G_1 from G by dropping out the arc $\alpha(ab)$, and form G_1' from G' by dropping out the arc $\alpha'(a'b')$, and letting the vertices a' and b' coalesce if they are not already the same vertex. Then G_1 and G_1' are duals, preserving the correspondence between their arcs.*

Let H_1 be any subgraph of G_1 and let H_1' be the complement of the corresponding subgraph of G_1'.

Case 1. Suppose the vertices a' and b' were distinct in G'. Let H be the subgraph of G identical with H_1. Then

$$n = n_1.$$

Let H' be the complement in G' of the subgraph corresponding to H. Then

$$r' = R' - n.$$

Now H' is the subgraph in G' corresponding to H_1' in G_1', except that H' contains the arc $\alpha'(a'b')$, which is not in H_1'. Thus if we drop out $\alpha'(a'b')$ from H' and let a' and b' coalesce, we form H_1'. In this operation, the number of connected pieces is unchanged, while the number of vertices is decreased by 1. Hence

$$r_1' = r' - 1.$$

As a special case of this equation, if H' contains all the arcs of G', we find

$$R_1' = R' - 1.$$

These equations give

$$r_1' = R_1' - n_1.$$

Thus G_1' is a dual of G_1.

Case 2. Suppose a' and b' are the same vertex in G'. In this case, defining H and H' as before, we form H_1' from H' by dropping out the arc $\alpha'(a'a')$. This leaves the number of vertices and the number of connected pieces unchanged. Thus two of the equations in Case 1 are replaced by the equations

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* Here and in a few other places we are using point-set theorems which, however, are geometrically evident.

$$r_1' = r_1, \quad R_1' = R_1.$$

The other equations are as before, so we find again that G_1' is a dual of G_1. The theorem is now proved.

Theorem 29. *A necessary and sufficient condition that a graph be planar is that it have a dual.*

We shall prove first the necessity of the condition. Given any planar graph G, we map it onto the surface of a sphere. If the nullity of G is N, it divides the sphere into $N+1$ regions. For let us construct G arc by arc. Each time we add an arc joining two separate pieces, the nullity and the number of regions remain the same. Each time we add an arc joining two vertices in the same connected pieces, the nullity and the number of regions are each increased by 1. To begin with, the nullity was 0 and the number of regions was 1. Therefore, at the end, the number of regions is $N+1$.

We construct G' as follows: In each region of the graph G we place a point, a vertex of G'. Therefore G' contains $V'=N+1$ vertices. Crossing each arc of G we place an arc, joining the vertices of G' lying in the two regions the arc of G separates (which may in particular be the same region, in which case this arc of G' is a 1-circuit). The arcs of G and G' are now in (1, 1) correspondence.

G' is the dual of G in the ordinary sense of the word. We must show it is the dual as we have defined the term.

Let us build up G arc by arc, removing the corresponding arc of G' each time we add an arc to G. To begin with, G contains no arcs and G' contains all its arcs, and at the end of the process, G contains all its arcs and G' contains no arcs. We shall show

(1) each time the nullity of G is increased by 1 upon adding an arc, the number of connected pieces in G' is reduced by 1 in removing the corresponding arc, and

(2) each time the nullity of G remains the same, the number of connected pieces in G' remains the same.

To prove (1) we note that the nullity of G is increased by 1 only when the arc we add joins two vertices in the same connected piece. Let ab be such an arc. As a and b were already connected by a chain, this chain together with ab forms a circuit P. Let $a'b'$ be the arc of G' corresponding to ab. Before we removed it, a' and b' were connected. Removing it, however, disconnects them. For suppose there were still a chain C' joining them. As a' and b' are on opposite sides of the circuit P, C' must cross P, by the Jordan Theorem, that is, an arc of C' must cross an arc of P. But we removed this arc of C' when we put in the arc of P it crosses. (1) is now proved.

The total increase in the nullity of G during the process is of course just N. Therefore the increase in the number of connected pieces in G' must be at least N. But G' was originally in at least one connected piece, and is at the end of the process in $V=N+1$ connected pieces. Thus the increase in the number of connected pieces in G' is just N (hence, in particular, G' itself is connected) and therefore this number increases only when the nullity of G increases, which proves (2).

Let now H be any subgraph of G, let H' be the complement of the corresponding subgraph of G', and let H' include all the vertices of G'. We build up H arc by arc, at the same time removing the corresponding arcs of G'. Thus when H is formed, H' also is formed. By (1) and (2), the increase in the number of connected pieces in forming H' from G' equals the nullity of H, that is,

$$p' - P' = n.$$

But

$$r' = V' - p', \quad R' = V' - P',$$

as G' and H' contain the same vertices. Therefore

$$r' = R' - n,$$

that is, G' is a dual of G.

To prove the sufficiency of the condition, we must show that if a graph has a dual, it is planar. It is enough to show this for non-separable graphs. For if the separable graph G has a dual, its components have duals, by Theorem 25, hence its components are planar, and hence G is planar, by Theorem 27. This part of the theorem is therefore a consequence of the following theorem:

THEOREM 30. *Let the non-separable graph G have a dual G'. Then we can map G and G' together on the surface of a sphere so that*

(1) *corresponding arcs in G and G' cross each other, and no other pair of arcs cross each other, and*

(2) *inside each region of one graph there is just one vertex of the other graph.*

The theorem is obviously true if G contains a single arc. (The dual of an arc ab is an arc $a'a'$, and the dual of an arc aa is an arc $a'b'$.) We shall assume it to be true if G contains fewer than E arcs, and shall prove it for any graph G containing E arcs. By Theorem 8, each vertex of G is on at least two arcs.

Case 1. G contains a vertex b on but two arcs, ab and bc. As G is non-separable, there is a circuit containing these arcs. Thus dropping out one of them will not alter the rank, while dropping out both reduces the rank by 1. As G' is a dual of G, the arcs corresponding to these two arcs are each of nullity 0, while the two arcs taken together are of nullity 1. They are thus of the form $\alpha'(a'b')$, $\beta'(a'b')$, the first corresponding to ab, and the second, to bc.

Form G_1 from G by dropping out the arc bc and letting the vertices b and c coalesce, and form G_1' from G' by dropping out the arc $\beta'(a'b')$. By Theorem 28, G_1 and G_1' are duals, preserving the correspondence between the arcs. As these graphs contain fewer than E arcs,* we can, by hypothesis, map them together on a sphere so that (1) and (2) hold; in particular, $\alpha'(a'b')$ crosses ac. Mark a point on the arc ac of G_1 lying between the vertex c and the point where the arc $\alpha'(a'b')$ of G' crosses it. Let this be the vertex b, dividing the arc ac into the two arcs ab and bc. Draw the arc $\beta'(a'b')$ crossing the arc bc. We have now reconstructed G and G', and they are mapped on a sphere so that (1) and (2) hold.

Case 2. Each vertex of G is on at least three arcs. As then G contains no suspended chain, and G is not a circuit and therefore is of nullity $N > 1$, we can, by Theorem 18, drop out an arc ab so that the resulting graph G_1 is non-separable. G' is non-separable, by Theorem 26, and hence the arc $a'b'$ corresponding to ab in G is not a 1-circuit. Drop it out and let the vertices a', b' coalesce into the vertex a_1', forming the graph G_1'. By Theorem 28, G_1 and G_1' are duals, and thus G_1' also is non-separable.

Consider the arcs of G' on a'. If we drop them out, the resulting graph G'' has a rank one less than that of G'. For if its rank were still less, G'' would be in at least three connected pieces, one of them being the vertex a'. Let c and d be vertices in two other connected pieces of G''. They are joined by no chain in G'', and hence every chain joining them in G' must pass through a', which contradicts Theorem 6. If we put back any arc, the rank is brought back to its original value, as a' is then joined to the rest of the graph. Hence, G' being a dual of G, the arcs of G corresponding to these arcs are together of nullity 1, while dropping out one of them reduces the nullity to 0. Therefore, by Theorem 9, these arcs form a circuit P. One of these arcs is the arc ab. The remaining arcs form a chain C. Similarly, the arcs of G corresponding to the arcs of G' on b' form a circuit Q, and this circuit minus the arc ab forms a chain D. C and D have the vertices a and b as end vertices. Also, the arcs of G_1 corresponding to the arcs of G_1' on a_1' form a circuit R. These arcs of G_1' are the arcs of G' on either a' or b', except for the arc $a'b'$ we dropped out. Thus the arcs of G_1 forming the circuit R are the arcs of the chains C and D.

As G_1 and G_1' contain fewer than E arcs, we can map them together on a sphere so that properties (1) and (2) hold. a_1' lies on one side of the circuit R, which we call the inside. Each arc of R is crossed by an arc on a_1', and thus there are no other arcs of G_1' crossing R. There is no part of G_1' lying inside R other than a_1', for it could have only this vertex in common with the rest of G_1', and G_1' would be separable. Also, there is no part of G_1 lying inside R, for any arc would have to be crossed by an arc of G_1', and any vertex would have to be joined to the rest of G_1 by an arc, as G_1 is non-separable.

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* Obviously G_1 is non-separable.

Let us now replace a_1' by the two vertices a' and b', and let those arcs abutting on a_1' that were formerly on a' be now on a', and those formerly on b', now on b'. As the first set of arcs all cross the chain C, and the second set all cross the chain D, we can do this in such a way that no two of the arcs cross each other. We may now join a and b by the arc ab, crossing none of these arcs. This divides the inside of R into two parts, in one of which a' lies, and in the other of which b' lies. We may therefore join a' and b' by the arc $a'b'$, crossing the arc ab. G and G' are now reconstructed, and are mapped on the sphere as required. This completes the proof of the theorem, and therefore of Theorem 29.

THEOREM 31. *A necessary and sufficient condition that a graph be planar is that it contain neither of the two following graphs as subgraphs:*

G_1. *This graph is formed by taking five vertices a, b, c, d, e, and joining each pair by an arc or suspended chain.*

G_2. *This graph is formed by taking two sets of three vertices, a, b, c, and d, e, f, and joining each vertex in one set to each vertex in the other set by an arc or suspended chain.*

This theorem has been proved by Kuratowski.* It would be of interest to show the equivalence of the conditions of the theorem and Theorem 29 directly, by combinatorial methods. We shall do part of this here, in the following theorem:†

THEOREM 32. *Neither of the graphs G_1 and G_2 has a dual.*

Suppose the graph G_1 had a dual. By Theorem 28, if G_1 contains a suspended chain, we can drop out one of its arcs and let the two end vertices coalesce, and the resulting graph will have a dual. Continuing, we see that the graph G_3, in which each pair of vertices of the set a, b, c, d, e are joined by an arc, must have a dual. Similarly, if G_2 has a dual, then the graph G_4, in which each vertex of the set a, b, c is joined to each vertex of the set d, e, f by an arc, must have a dual. Both of these are impossible.

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* Fundamenta Mathematicae, vol. 15 (1930), pp. 271–283.

† The other half has recently been proved by the author. See Bulletin of the American Mathematical Society, abstract (38-1-39). (Note added in proof.)

(a) *The graph* G_3. To avoid subscripts, let us call it G. Suppose it had a dual, G'. Then

$$R = N' = 4,$$

$$N = R' = 6,$$

$$E = E' = 10.$$

If G' has isolated vertices, we drop them out, which does not alter its relation to G.

(1) There are no 1-circuits, 2-circuits or triangles in G'. For if there were, dropping out the corresponding arcs of G would have to reduce the rank of G. But we cannot reduce its rank without dropping out at least four arcs.

(2) G' contains at least five quadrilaterals. For if we drop out the four arcs on any vertex of G, the rank is reduced by 1, and if we put back any of these arcs, the rank is brought back to its original value; Theorem 9 now applies.

(3) At least two of these quadrilaterals have an arc in common, as there are but ten arcs in G'.

There are just two ways of forming two quadrilaterals out of fewer than eight arcs without forming any 2-circuits or triangles. One of these graphs, I_1', contains the arcs $a'b'$, $b'e'$, $a'c'$, $c'e'$, $a'd'$, $d'e'$. The other, I_2', contains the arcs $a'e'$, $e'f'$, $f'b'$, $b'a'$, $e'c'$, $c'd'$, $d'f'$. But there is no subgraph of the type I_1' in G', for this subgraph is of rank 4 and nullity 2, and there would have to be a subgraph of G of rank 2 and nullity 2, and such a graph contains a 1- or a 2-circuit, of which there are none in G. Hence G' contains a subgraph I_2'.

(4) Each vertex of G' is on at least three arcs, as there are no 1- or 2-circuits in G.

Each of the vertices a', b', c', d' of I_2' is on but two arcs. Hence there must be another arc on each of these vertices. As I_2' contains seven arcs, and G' contains but ten, one of the three arcs left must join two of these vertices. But if we add an arc $a'b'$ or $c'd'$, we would form a 2-circuit; if we add an arc $a'c'$ or $b'd'$, we would form a triangle; if we add an arc $a'd'$ or $b'c'$, we would form a graph of the type I_1'. As G' contains none of these graphs, we have a contradiction.

(b) *The graph* G_4. Let us call it G. If it has a dual G', then

$$R = N' = 5,$$

$$N = R' = 4,$$

$$E = E' = 9.$$

We proceed exactly as for the graph G_3. In outline:

(1) G' contains no 1- or 2-circuits.

(2) There is no subgraph of G' containing four vertices, each pair being joined by an arc. For this graph is of rank 3 and nullity 3, and G would have to contain a subgraph of rank 2 and nullity 1, that is, a 2-circuit.

(3) There are at least nine subgraphs of G' of rank 3 and nullity 2, and hence of the form $a'b', a'c', b'c', b'd', c'd'$, as there are nine quadrilaterals in G.

(4) As G' contains but nine arcs, two of these subgraphs have an arc in common. There is therefore a subgraph of one of the forms $I_1' : a'e', a'b', b'e', a'c', c'e', a'd', d'e'$, or $I_2' : a'e', a'b', b'e', b'c', c'e', c'd', d'e'$.

(5) Each vertex of G' is on at least four arcs.

Now each of the graphs I_1', I_2' contains seven arcs. We have but two arcs left which we must place so that each vertex of I_1' or I_2' is on at least four arcs. This cannot be done. The theorem is now proved.

Theorem 31 together with this theorem gives an alternative proof of the second part of Theorem 29. For suppose a graph G had a dual. Then it contains neither the graph G_1 nor G_2. For if it did, dropping out all the arcs of G but those forming one of these graphs, Theorem 28 tells us that this graph has a dual. But we have just seen that this is not so. Hence, by Theorem 31, G is planar.

Euler's formula. Map any connected planar graph G on a sphere, and construct its connected dual G' as described in the proof of Theorem 29. Then in each region of G there is a vertex of G'. Let F be the number of regions (or faces) in G. Then

$$R' = N,$$
$$R = V - 1,$$
$$R' = V' - 1,$$
$$V' = F,$$

and hence

$$\begin{aligned} V - E + F &= R + 1 - E + N + 1 \\ &= 2, \end{aligned}$$

which is Euler's formula.

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A CHARACTERIZATION OF THE CLOSED 2-CELL[*]

BY

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1. Introduction. A number of characterizations have been given of the simple closed surface.[‡] The proofs involve considerable point set difficulties. We give here a characterization of the closed 2-cell, that is, a point set homeomorphic with a circle and its interior. The fundamental theorem is partly of a combinatorial and partly of a continuity nature. It reads

THEOREM I. *Let R be a continuous curve*[§] *containing the simple closed curve J, such that*

(1) *J is irreducibly homologous to zero in R, and*

(2) *If γ is an arc with just its two end points a and b on J, then $R-\gamma$ is not connected.*

Let R' and J' be defined similarly. Then R and R' are homeomorphic, with J corresponding with J'.

That R is a closed 2-cell then follows immediately from the following theorem. We note that J corresponds with the circle, that is, J is the boundary of R.

THEOREM II. *If I is a circle in the plane and S is I with its interior, then S and I satisfy the conditions prescribed for R and J in the above theorem.*

The exact meaning of Condition (1) of Theorem I is given in §4; a stronger condition is the following: For every $\epsilon>0$ and any two points a and b on J, there is a set of points a_{ij} in R, $1\leqq i\leqq m$, $1\leqq j\leqq n$, such that all points a_{1j} coincide with a, all points a_{mj} coincide with b, all points a_{i1} lie on one arc ab of J, all points a_{in} lie on the other arc ab of J, and[||]

$$\rho(a_{ij}, a_{i+1,j}) < \epsilon, \quad \rho(a_{ij}, a_{i,j+1}) < \epsilon;$$

moreover, this does not hold in any proper subset of R containing J.

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[*] Presented to the Society, October 31, 1931; received by the editors April 13, 1932.

[†] National Research Fellow.

[‡] That is, a point set homeomorphic with the surface of a sphere. See L. Zippin, American Journal of Mathematics, vol. 52 (1931), pp. 331–350; these Transactions, vol. 31 (1929), pp. 744–770; C. Kuratowski, Fundamenta Mathematicae, vol. 13 (1929), pp. 307–318; also references in these papers.

[§] See Lemma A.

[||] $\rho(p, q)$ = distance from p to q, or in general, distance between two point sets; $\delta(S)$ = diameter of S; $V_\epsilon(S)$ = those points p for which $\rho(p, S)<\epsilon$; $W_\epsilon(S)$ = those points p for which $\rho(p, S)\leqq\epsilon$.

Notations and preliminary theorems are given in §§2, 3 and 4; an outline of the proof of Theorem I will be found in §5. The Jordan and related theorems follow of course from the above theorems.

2. **Point set background.** Elementary properties of point sets we shall need may be found in Hausdorff, *Mengenlehre*, chapter VI. A continuous curve is a metric space which can be expressed as the continuous image of a closed line segment. An arc is the topological image of a closed line segment; a simple closed curve, the topological image of a circle.

Two fundamental lemmas are the following:

LEMMA A.* *A compact, connected and locally connected metric space is a continuous curve, and conversely.*

LEMMA B.† *A continuous curve is arcwise connected.*

That is, any two points p and q in the set are end points of an arc pq in the set. Using the definition of a continuous curve, it is easily seen that two continuous curves which have common points form a continuous curve.

From these lemmas we deduce the following known theorems.

LEMMA C. *Any continuum C of diameter $<\epsilon$ in a continuous curve R is contained in a continuous curve C' in R of diameter $<\epsilon$.*

Say $\delta(C)=\epsilon-\epsilon'$. R being the continuous image of a closed line segment, we can divide this segment into segments so small that the diameter of the image of each is $<\epsilon'$. We let C' be the union of all of these images which have points in common with C.

LEMMA D. *A continuous curve R is locally arcwise connected.*

That is, given a point p and an $\epsilon>0$, there is a $\delta>0$ such that if $q\subset V_\delta(p)$, then there is an arc pq in R of diameter $<\epsilon$. As R is locally connected, we can take δ so that if $q\subset V_\delta(p)$, there is a continuum C in R of diameter $<\epsilon$ containing p and q. The continuum C is contained in a continuous curve C' of diameter $<\epsilon$, and C' is arcwise connected; hence there is an arc $pq\subset C'\subset R$, and $\delta(pq)<\epsilon$.

R is of course uniformly locally arcwise connected, by the Borel Theorem.

LEMMA E. *A connected open subset R' of a continuous curve R is arcwise connected.*

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* See G. T. Whyburn, *Concerning continuous images of the interval*, American Journal of Mathematics, vol. 53 (1931), pp. 670–674.

† See references in R. L. Moore, *Report on continuous curves*, Bulletin of the American Mathematical Society, vol. 29 (1923), p. 293, footnote (†).

If there are two points p and q in R' which are joined by no arc in R', let A contain p and all points of R' joined to p by an arc in R', and put $B=R'-A$; then there is no arc joining a point of A to a point of B in R'. As R' is connected, there is a point p' in one of these sets, say B, which is a limit point of points of the other set, A. As R' is open in R, $\rho(p', R-R')=\epsilon>0$. We can take q' in A so close to p' that there is an arc $p'q'$ in R of diameter $<\epsilon$. But then $p'q'\subset R'$, a contradiction.

Suppose R is connected, and $p\subset R$ is such a point that $R-p$ is not connected. Then p is called a cut point of R.

Lemma F. *Let R be a continuous curve without a cut point. Then for every $\epsilon>0$ there is a $\delta>0$ such that if $\rho(q, p)\geqq\epsilon$ and $\rho(q', p)\geqq\epsilon$, then there is an arc qq' with no points in $V_\delta(p)$.*

Suppose the contrary. Then there are three sequences of points $\{p_n\}$, $\{q_n\}$, $\{q_n'\}$, approaching points p, q, q', respectively, with $\rho(q_n, p_n)\geqq\epsilon$, $\rho(q_n', p_n)\geqq\epsilon$, and such that for each n, any arc q_nq_n' must contain points in $V_{\delta_n}(p_n)$, where $\lim_{n\to\infty}\delta_n=0$. By Lemma D it is seen that for any n greater than some N there are arcs q_nq, $q_n'q'$, with no points in $V_{\epsilon/2}(p)$. It follows that any arc qq' must pass through p, contradicting Lemma E (as p is not a cut point).

3. Combinatorial background.* A *k-simplex*, or *abstract k-simplex*, is a set of k elements (say points) $a_1a_2\cdots a_k$. The order in which we write the points is immaterial. For $k=0$, 1 and 2 we use also the terms *vertex*, *segment* and *triangle* respectively. A *k-chain* is a set of k-simplexes, and is written as the sum of these simplexes. The *sum* (mod 2) of several k-chains is the k-chain containing those simplexes which occur in an odd number of the k-chains.

The *boundary K* of a *k-simplex L, $k>0$*, is the sum of all $(k-1)$-simplexes formed by dropping out one of the vertices of the simplex. We write $L\rightarrow K$. A 0-simplex has no boundary. Thus

$$a\rightarrow 0,\ ab\rightarrow a+b,\ abc\rightarrow ab+ac+bc.$$

The *boundary* of a k-chain is the sum (mod 2) of the boundaries of the simplexes of the chain. Thus

$$ab+bc+cd\rightarrow a+d,\ abc+bcd\rightarrow ab+ac+bd+cd.$$

Evidently *the boundary of a sum of several k-chains is the sum of the boundaries of the chains.* If a k-chain has no boundary, it is called a k-cycle. (Any 0-chain is a 0-cycle.) *The boundary of a k-chain ($k>0$) is a $(k-1)$-cycle.* This is evi-

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[*] Compare L. Vietoris, *Über den höheren Zusammenhang kompakter Räume*, Mathematische Annalen, vol. 97 (1927), pp. 454–472.

dent if the k-chain is a k-simplex. The general case then follows from the last theorem.

LEMMA G. *If* $K\rightarrow a+b$ *is a* 1-*chain, then there is a chain of segments* aa_1, a_1a_2, $\cdots$, a_nb *in* K.

For otherwise we could divide the segments of K into two groups $K_1\supset a$ and $K_2\supset b$, no two simplexes from different groups having a common vertex. But then $K_1\rightarrow a$, $K_2\rightarrow b$, which cannot be, as the boundary of any 1-chain contains an even number of vertices.

A 1-circuit is a 1-cycle of the form a_1a_2, a_2a_3, $\cdots$, $a_{n-1}a_n$, a_na_1, the vertices being distinct except as shown.

LEMMA H. *Any* 1-*cycle* K *is a sum of* 1-*circuits.*

If a_1a_n is a segment of K, then $K+a_1a_n\rightarrow a_1+a_n$, as $K\rightarrow 0$ and $a_1a_n\rightarrow a_1+a_n$. We can thus find a set of distinct segments and vertices a_1a_2, $\cdots$, $a_{n-1}a_n$ in $K+a_1a_n$ not containing a_1a_n. This with a_1a_n is a 1-circuit K_1. As $K_1\rightarrow 0$, $K+K_1$ is a 1-cycle containing no segments of K_1, and it contains a 1-circuit K_2. Continuing, we find $K=K_1+K_2+\cdots+K_m$.

4. A k-chain K is said to *lie in a point set* R if each vertex of K is in R. Any vertex now has both a name and a position. Two vertices are distinct if their names are distinct, irrespective of whether they coincide in position or not. ϵ being a positive number, a k-simplex $K\subset R$ is called an (ϵ, k)-*simplex* in R if $\delta(K)<\epsilon$, i.e. if any two vertices of K are within ϵ of each other. A k-chain is an (ϵ, k)-*chain* if each of its simplexes is an (ϵ, k)-simplex. A k-cycle K in S is said to be ϵ-*homologous to zero* ($K\epsilon\sim 0$) in R if there is an $(\epsilon, k+1)$-chain L in R of which K is the boundary. *If* $K_1\epsilon\sim 0$ *and* $K_2\ \epsilon\sim 0$, *then* $K_1+K_2\ \epsilon\sim 0$. We write also $K_1\epsilon\sim K_2$ for $K_1+K_2\epsilon\sim 0$. *If* $K_1\epsilon\sim K_2$ *and* $K_2\epsilon\sim K_3$, *then* $K_1\epsilon\sim K_3$.

Suppose the closed set R contains the simple closed curve J. If for every $\epsilon>0$ there is a $\delta>0$ such that any $(\delta, 1)$-cycle on J is $\epsilon\sim 0$ in R, then we say that $J\sim 0$ *in* R. If J is ~ 0 in R but is not ~ 0 in any proper closed subset of R containing J, then we say that J is *irreducibly* ~ 0 in R.

LEMMA I. *Given a simple closed curve* J, *let us divide it into the arcs** $\overline{a_1a_2}$, $\overline{a_2a_3}$, $\cdots$, $\overline{a_{n-1}a_n}$, $\overline{a_na_1}$, *each of diameter* $<\epsilon/2$. *Let* δ *be smaller than the distance between any two of these arcs which have no common points. Then if* $K'=a_1a_2+a_2a_3+\cdots+a_na_1$ *and* K *is any* $(\delta, 1)$-*cycle on* J, K *is either* $\epsilon\sim 0$ *or* $\epsilon\sim K'$ *on* J.

By Lemma H, K is a sum of 1-circuits K_1, $\cdots$, K_m. If we show that each

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* Here, $\overline{a_1a_2}$ denotes an arc, and a_1a_2, a segment.

K_i is $\epsilon\sim\alpha_iK'$, $\alpha_i=0$ or 1, it will follow that $K=\sum K_i\ \epsilon\sim\sum\alpha_iK'=0$ or K' (depending on whether $\sum\alpha_i$ is even or odd), and the lemma will be proved.

Consider any $K_i=b_1b_2+b_2b_3+\cdots+b_sb_1$, say. If a vertex b_j of K_i does not lie on any point a_k, say $b_j\subset\overline{a_ka_{k+1}}$; add to K_i the boundary of the ϵ-triangles $b_{j-1}b_ja_k'+b_jb_{j+1}a_k'$, where a_k' is a new vertex lying on a_k. The result is an $(\epsilon, 1)$-circuit $K_i^{(1)}\,\epsilon\sim K_i$, the vertex b_j having been replaced by the vertex a_k'. Repeat the process till we have an $(\epsilon, 1)$-circuit $K''=c_1c_2+c_2c_3+\cdots+c_sc_1\ \epsilon\sim K_i$.

Now any two consecutive vertices c_j, c_{j+1} lie on the same or consecutive vertices of K'. Suppose c_j is on a_k and c_{j+2} is on a_{k+p}, $p\neq 2$ or -2. Then add the boundary of $c_jc_{j+1}c_{j+2}$, replacing the segments $c_jc_{j+1}+c_{j+1}c_{j+2}$ by the single segment c_jc_{j+2}. Continue till we arrive at a (possibly void) $(\epsilon, 1)$-circuit $K^*=d_1d_2+\cdots+d_rd_1\,\epsilon\sim K_i$. If d_1 lies on a_k, then d_{j+1} lies on $a_{k\pm j}$, where we put $n+p=p$, etc.

If K^* contains no segments, $K_i\ \epsilon\sim 0$. Otherwise, following the vertices $d_1, d_2, \cdots, d_r, d_1$ of K^*, we have gone around J p times say. Add to K^* the boundaries of all the $2r$ ϵ-triangles of the following sort. If d_j lies on a_k, and d_{j+1} on $a_{k\pm1}$, two of the triangles are $d_jd_{j+1}a_k$ and $d_{j+1}\,a_ka_{k\pm1}$. The result is an $(\epsilon, 1)$-cycle $pK'=0$ or K'. Thus $K_i\,\epsilon\sim 0$ or K', and the proof is complete.

An immediate consequence of this lemma is

Lemma J. *Let the simple closed curve J lie in the closed set R. If for every $\epsilon>0$ there is a 1-cycle K' in J as above described which is $\epsilon\sim 0$ in R, then $J\sim 0$ in R.*

Lemma K. *If γ is an arc, then for every $\epsilon>0$ there is a $\delta>0$ such that any $(\delta, 1)$-cycle on γ is $\epsilon\sim 0$ on γ.*

The proof below holds in fact if γ is a closed k-cell, any k. It is sufficient to prove it for the case that γ is a closed line segment, in which case we can take $\delta=\epsilon/2$.[†]

Let K be a $(\delta, 1)$-cycle on γ, let a_0b_0 be a segment of K, and say $\delta(\gamma)=\alpha$. Choose a fixed point p in γ, and an integer $n>\alpha/\delta$. Let the vertices $a_1, a_2, \cdots, a_{n-1}$ divide the segment a_0p into n equal parts, and similarly for the vertices $b_1, b_2, \cdots, b_{n-1}$. Add to K the boundaries of all triangles of the form $a_ia_{i+1}b_i$, $a_{i+1}b_ib_{i+1}$, $a_{n-1}b_{n-1}p$, and of all similar triangles corresponding to the other segments of K. The result is 0. As all the triangles employed are ϵ-triangles, $K\,\epsilon\sim 0$ in γ.

5. Outline of the proof of Theorem I. The proof runs as follows.

[†] The essential point in the proof below is that γ is convex: any two points of γ are end points of a line segment in γ. The proof is then easily extended to the case of any set homeomorphic with γ.

(a) In §6 we show how an arc γ can be drawn in R crossing J,† avoiding two given closed sets. $R-\gamma$ is not connected.

(b) In §7 we prove some lemmas. These show (§8) that $R-\gamma$ contains exactly two components A' and B'. If $A=A'+\gamma$, then A and its boundary curve J_A (which is γ plus a part of J) satisfy condition (1) of the theorem; similarly for $B=B'+\gamma$ and J_B. Further, A and B are continuous curves.

(c) In §9 it is shown that any arc in A (or B) crossing J_A (J_B) divides A (B). Thus A and J_A (B and J_B) satisfy all the conditions of the theorem. Hence we can cut up each set just as we cut up R, and can continue indefinitely.

(d) The object of §10 is to prove that R may be cut into pieces of arbitrarily small diameter.

(e) The homeomorphism between R and R' is now easily established. We cut R up indefinitely, and cut R' in a corresponding fashion. Any point p of R lies in a descending sequence of pieces; the corresponding sequence in R' determines a point p', which we let correspond to p.

We turn now to the detailed proof.

6. An arc crossing J. We prove here

Lemma L.‡ *Let the simple closed curve J be ~ 0 in the continuous curve R. Let c and d be two points of J, dividing J into the two arcs η_1 and η_2. If C and D are two closed sets in R containing c and d respectively, and $C\cdot D=0$, then there is an arc γ in R joining η_1 to η_2 which has no points in C or in D.*

Say $\rho(C, D)=3\epsilon$, and put $C'=W_\epsilon(C)$, $D'=W_\epsilon(D)$; then $\rho(C', D')=\epsilon$. Take σ so small that any two points in R within σ of each other are joined by an arc of diameter $<\epsilon$ (Lemma D). Take δ so small that any $(\delta, 1)$-cycle on J is $\sigma\sim 0$ in R. Construct the $(\delta, 1)$-cycle $K=cc_1+c_1c_2+\cdots+c_md+dd_1+d_1d_2+\cdots+d_nc$, $c_i\subset\eta_1$, $d_i\subset\eta_2$. There is a $(\sigma, 2)$-chain

$$L = L_C + L_D \rightarrow K$$

in R, where we let L_C contain all those triangles of L with vertices in C', and let L_D be the rest of L.

Say

$$L_C \rightarrow K_C = K_C' + K^*,$$

where we let K_C' contain all those segments of K_C which are also in K. As $L_C\subset V_\sigma(C')$, $K^*\cdot D'=0$. Define K_D' by the relation

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† That is, γ lies in R, and has only its end points on J.

‡ Compare P. Urysohn, *Über Räume mit verschwindender erster Brouwerscher Zahl*, Proceedings, Amsterdam Akademie van Wetenschappen, vol. 31 (1928), pp. 808–810.

$$L_D \rightarrow K_D = K_D' + K^*.$$

Adding these relations gives L on the left, and hence K on the right:

$$K = K_C' + K_D'.$$

As all the segments of K_C' are in K, K_D' must contain just those segments of K not in K_C'; in particular, it contains no segments of K^*. Hence all the segments of K^* are present in $K_D' + K^*$, the boundary of L_D (i.e. none have canceled out with segments of K_D'). Hence, as $L_D \cdot C' = 0$,

$$K^* \cdot C' = K^* \cdot D' = 0.$$

As K_C is the boundary of L_C, it is a 1-cycle; hence

$$K_C + cc_1 = K_C' + K^* + cc_1 \rightarrow c + c_1.$$

By Lemma G, $K_C + cc_1$ contains a chain of segments joining c_1 to c. Following this chain, let p_s be the first vertex in η_2, and p_0, the last vertex before p_s in η_1, and say $p_0p_1, p_1p_2, \cdots, p_{s-1}p_s$ are the segments in between. We shall show that these segments are in K^*. If $s > 1$ this is obvious, as then $p_1, \cdots p_{s-1}$ exist and are not on J. Suppose $s = 1$ and p_0p_1 is not in K^*; then it is in $K_C' + cc_1$. It could only be the segment cc_1. But cc_1 lies in K and not in K_D, hence it is in K_C; it is not in K^*, hence it is in $K_C + K^* = K_C'$, and therefore not in $K_C' + cc_1$. This proves the statement.

Now let $\overline{p_ip_{i+1}}$ be an arc of diameter $< \epsilon$ in R, $i = 0, \cdots, s-1$. These arcs form a continuous curve, from which we can pick out an arc γ (Lemma B) joining η_1 to η_2; we can take γ so only its end points are on J. As $p_ip_{i+1} \subset K^*$ and $\delta(\overline{p_ip_{i+1}}) < \epsilon$, γ has no points in C or in D, and the lemma is proved.

7. We prove three lemmas.

LEMMA M. *If $J \subset C$, $J \sim 0$ in $C + D$, and $C \cdot D =$ an arc γ, then $J \sim 0$ in C.*

Given an $\epsilon > 0$, choose first ϵ_1 so small that any $(3\epsilon_1, 1)$-cycle on γ is $\epsilon \sim 0$ in γ (Lemma K). Take next $\epsilon_2 < \epsilon_1$ so that if $p \subset D$ and $\rho(p, C) < \epsilon_2$, then $\rho(p, \gamma) < \epsilon_1$. (If $D_1 = D - D \cdot V_{\epsilon_1}(\gamma)$, take $\epsilon_2 < \rho(D_1, C)$.) Take finally $\delta < \epsilon_2$ so that any $(\delta, 1)$-cycle K on J is $\epsilon_2 \sim 0$ in $C + D$; we shall show that $K \epsilon \sim 0$ in C.

Let $L \rightarrow K$ be an $(\epsilon_2, 2)$-chain in $C + D$. Take any vertex p of L in $D \cdot V_{\epsilon_2}(C) - \gamma$, and replace it by a vertex $p' \subset \gamma$, where $\rho(p, p') < \epsilon_1$. L is thus replaced by a $(3\epsilon_1, 2)$-chain L', in which each triangle lies wholly in either C or D. Moreover, $L' \rightarrow K$, as no vertices of K have been moved.

Put $L' = L_C + L_D$, where L_C contains those triangles of L' in C. Say

$$L_C \rightarrow K + K^*; \text{ then } L_D \rightarrow K^*.$$

K^* is a $(3\epsilon_1, 1)$-cycle lying in $C \cdot D = \gamma$; it bounds an $(\epsilon, 2)$-chain L^* in γ. Hence

$$L_C + L^* \to (K + K^*) + K^* = K.$$

L_C+L^* is an $(\epsilon, 2)$-chain in C, and the lemma is proved.

Lemma N. *Let $A \cdot B=\gamma$, an arc whose end points are a and b. Let the arcs α and β join a and b in A and B respectively, neither having any points other than a and b in common with γ. If $\alpha+\beta\sim 0$ in $A+B$, then $\alpha+\gamma\sim 0$ in A.*

Given an $\epsilon>0$, choose ϵ_1, ϵ_2 and δ as in the last lemma. Take $(\delta, 1)$-chains K_α, K_β and K_γ in α, β and γ respectively, each bounded by $a+b$; by Lemma J, it is sufficient to show that $K_\alpha+K_\gamma\ \epsilon\!\sim 0$ in A.

$K_\alpha+K_\beta$ bounds an $(\epsilon_2, 2)$-chain L in $A+B$; we move each vertex of L in $B\cdot V_{\epsilon_2}(A)-\gamma$ onto γ, giving a $(3\epsilon_1, 2)$-chain $L'\to K_\alpha+K'_\beta$. Say $L'=L_A+L_B$, where $L_A\subset A$, $L_B\subset B$. If $L_A\to K_\alpha+K^*$, then $L_B\to K'_\beta+K^*$, and $K^*\subset\gamma$. K^*+K_γ is a $(3\epsilon_1, 1)$-cycle on γ bounding an $(\epsilon, 2)$-chain L^* in γ. Hence $L_A+L^*\to K_\alpha+K_\gamma$ in A, completing the proof.

Lemma O. *Let α, β and γ be three arcs such that $\alpha\cdot\beta=\alpha\cdot\gamma=\beta\cdot\gamma=a+b$. Say $\alpha+\gamma\subset A$ and $\beta+\gamma\subset B$. If $\alpha+\gamma\sim 0$ in A and $\beta+\gamma\sim 0$ in B, then $\alpha+\beta\sim 0$ in $A+B$.*

Define K_α, K_β, K_γ as before; we need merely show that $K_\alpha+K_\beta\ \epsilon\!\sim 0$ in $A+B$. There are $(\epsilon, 2)$-chains L_A and L_B such that $L_A\to K_\alpha+K_\gamma$ in A and $L_B\to K_\beta+K_\gamma$ in B; hence $L_A+L_B\to K_\alpha+K_\beta$ in $A+B$.

8. The set $R-\gamma$. Let γ be any arc in R crossing J; say the end points of γ divide J into the two arcs α and β. By condition (2) of the theorem, $R-\gamma$ is not connected. Let A' and B' be those components of $R-\gamma$ containing $<\alpha>$† and $<\beta>$ respectively. These are not the same component. For if they were, putting $A = A'+\gamma, D = R - A'$, we have $J\subset A$, $J\sim 0$ in $R=A+D$, and $A\cdot D=\gamma$; hence, by Lemma M, $J\sim 0$ in A, a proper subset of R, contrary to condition (1) of the theorem.

The same reasoning shows that R has no cut point p; we need merely replace γ by p in Lemma M and above.

Put

$$A = A' + \gamma,\ B = B' + \gamma.$$

If $D=R-A'$, then $A\cdot D=\gamma$ and $J=\alpha+\beta\sim 0$ in $R=A+D$. Hence, by Lemma N, $\alpha+\gamma\sim 0$ in A. Similarly, $\beta+\gamma\sim 0$ in B. Consequently, by Lemma O, $J\sim 0$ in $A+B$, from which follows that $A+B=R$.

Moreover, $\alpha+\gamma$ is irreducibly ~ 0 in A. For if $\alpha+\gamma\sim 0$ in A^*, $\alpha+\gamma\subset A^*\subset A$, then, by Lemma O, $\alpha+\beta\sim 0$ in A^*+B; hence $A^*+B=R$, which is only possible if $A^*=A$. Similarly, $\beta+\gamma$ is irreducibly ~ 0 in B.

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† $<\alpha>$ is α except for its end points, etc.

Let us show that A is a continuous curve. It is connected, as A' is; it is self-compact, being a closed subset of a compact space. A is locally connected. For if p and q are points of A close enough together, there is an arc pq in R of small diameter; if pq lies partly in B', we can replace that part of it by an arc of γ of small diameter. Lemma A now applies. Similarly, B is a continuous curve.

9. We shall now show that any arc δ crossing $J_A=\alpha+\gamma$ in A divides A. The following two lemmas will be useful.

LEMMA P. *If η_1 and η_2 are arcs contained within the arcs γ and β respectively, then there is an arc pq crossing $J_B=\beta+\gamma$ in B, with $p \subset \eta_1$, $q \subset \eta_2$.*

This is an immediate consequence of Lemma L, if we take, for the closed sets of that lemma, the closed intervals of J_B complementary to η_1 and η_2.

LEMMA Q. *There are no two arcs ab and cd in R without common points, each crossing J, whose end points are in the order $acbd$ on J.*

This follows directly from what we have seen above.

To show that δ divides A, we must consider four cases.

Case 1. Both end points of δ lie on α. Suppose $A-\delta$ is connected; then it is arcwise connected, by Lemma E. Hence there is an arc in $A-\delta$ joining a point p of α lying between the two end points of δ and a point q within γ. If η_1 is an arc within γ containing q, there is an arc rs in B joining η_1 to a point s within β, with only its end points r and s on J_B, by Lemma P. The arc $pqrs$ crosses J and does not touch δ. But the end points of this arc alternate with those of δ on J, contradicting Lemma Q.

Case 2. δ is an arc cd, where c lies within α, d lies within γ. If $A-\delta$ is connected, let pq be an arc in this set joining points of α on opposite sides of c. If η_1 is an arc of γ containing d but not touching pq, let the arc rs join η_1 to β in B; then the arcs pq and $cdrs$ contradict Lemma Q.

Case 3. The end points c and d of δ lie within $\gamma=ab$, say in the order $acdb$. If $A-\delta$ is connected, let pq be an arc in this set joining a point p within α to a point q in γ between c and d. If η_1 is an arc of γ containing q but not touching δ, let r_1s_1 be an arc in B joining η_1 to a point s_1 within β.

The arcs acr_1 of γ and r_1s_1 form an arc acr_1s_1 crossing J; hence

$$R - acr_1s_1 = C_1 + C_2,$$

where C_1 contains the open arc $<as_1>$ of β, and C_2 contains b and points connected with b. As r_1s_1 lies in B, $A' \subset C_1+C_2$; the connected set $A'+b$ lies thus in C_2. If η_2 is an arc of γ containing c but not touching η_1, and r_2s_2 is an arc in C_1 joining η_2 to a point s_2 of β between a and s_1, then $\eta_2+r_2s_2$ does not touch pqr_1s_1, and has only the point c in common with δ.

Similarly, if η_3 is an arc of γ containing d but not touching η_1, there is an arc r_3s_3 in $R-bdr_1s_1$ such that r_3 lies in η_3, s_3 lies in β between s_1 and b, and $\eta_3+r_3s_3$ does not touch pqr_1s_1 and has only the point d in common with δ. The arc r_3s_3 does not touch r_2s_2, as it lies in C_2. Thus the two arcs pqr_1s_1 and $s_2r_2cdr_3s_3$ $(cd=\delta)$ contradict Lemma Q.

Case 4. The same as Case 3, except that $c=a$ or $d=b$, say the latter. Then, in the notation of Case 3, the arcs pqr_1s_1 and s_2r_2cb $(cb=\delta)$ contradict Lemma Q.

This completes the proof that A and J_A (B and J_B) satisfy the conditions of Theorem I.

10. **The cutting up of** R. We are concerned with the following lemma.

LEMMA R. *R may be cut into a finite number of pieces of arbitrarily small diameter.*

Given an $\epsilon>0$, choose $\delta<\epsilon$ so as to satisfy the requirement in Lemma F. Suppose R is cut up so that the diameter of the boundary of each piece is $<\delta$. Then each piece is of diameter $<3\epsilon$. For otherwise there is a point q of some piece R_i at a distance $\geqq\epsilon$ from its boundary J_i. Let p be a point of J_i, and q', a point of $R-R_i$ at a distance $\geqq\epsilon$ from p. Every arc from q to q' must cut the boundary J_i of R_i and thus must pass within δ of p, contradicting Lemma F.

The lemma thus follows from

LEMMA S. *Given a $\delta>0$, R can be cut up so that the diameter of the boundary of each piece is $<\delta$.*

Express R as the union of a finite number of continua:

$$R = K_1 + K_2 + \cdots + K_m,\ \delta(K_i) < \delta/2.$$

We shall cut up R in such a manner that no two of these continua K_i and K_j have points on the boundary of the same piece of R, if $K_i\cdot K_j=0$; the lemma will then follow.

Suppose we have cut R up a certain amount (perhaps not yet at all), into the pieces $R_1, R_2, \cdots, R_n$, with boundaries $J_1, J_2, \cdots, J_n$ (we may have R and J alone). Of course *each boundary J_i separates R_i from the rest of R.* Take any two continua, say K_1 and K_2, with $K_1\cdot K_2=0$, each of which has points on one of these J_i, say J_1. We shall cut R up further so that in the new pieces there is no one (i.e. no *piece*, not merely no *boundary* of a piece) which has any points in common with both K_1 and K_2; then on any further cutting up of R, this will still be true.

Divide the points of J_1 into three sets, as follows. We put a point x into the first set if it lies in K_1, or if following J_1 in both directions we reach points of K_1 before reaching points of K_2; we put x into the second set if the same conditions hold with K_1 and K_2 interchanged; all other points we put into the third set. This set L_3' consists of open intervals of J_1, each being bounded by a point of K_1 on one end and a point of K_2 on the other. The points of the first set together with the points $K_1 \cdot R_1$ form a closed set L_1, and those of the second set together with $K_2 \cdot R_1$ form a closed set L_2. Then $\rho(L_1, L_2) > 0$ as $L_1 \cdot L_2 = 0$, from which follows that there are but a finite number of intervals in L_3'. As K_1 is connected, each component of L_1 has points on J_1, and thus on one of the intervals L_3 of J_1 complementary to the intervals of L_3'. Thus there are a finite number of components $L_{11}, L_{12}, \cdots, L_{1m_1}$ in L_1. Similarly there are a finite number of components $L_{21}, L_{22}, \cdots, L_{2m_2}$ in L_2.

We shall now cut R_1 into a number of pieces, in each of which either K_1 has no points or K_2 has no points. Suppose $L_{31}', \cdots, L_{3m_3}'$ and $L_{31}, \cdots, L_{3m_3}$ are the intervals of L_3' and L_3 respectively, and say they lie in the order L_{31}, L_{31}', L_{32}, L_{32}', $\cdots$, L_{3m_3}, L_{3m_3}' on J_1. If we go around J_1, the intervals of L_3 lie alternately in L_1 and L_2. Starting at L_{31}, which lies in L_{11} say, go around J_1 till we reach another interval L_{3k} in L_{11} (we may have gotten back to L_{31}). Put L_{32}, $L_{3,k-1}$ and all of J_1 between these into a set M_2' (which may be L_{32} alone), and put L_{3k}, L_{31}, and all of J_1 between these on the other side from L_{32} into a set M_1' (which may be L_{31} alone). L_{31}' and $L_{3,k-1}'$ are the two intervals of J_1 complementary to M_1' and M_2'.

No set L_{1i} or L_{2j} has points in both M_1' and M_2'. This follows for L_{11} by construction. If it were false for some other set, say L_{1s}, then L_{1s} would have points on two intervals L_{3p} and L_{3q} separated by L_{31} and L_{3k} on J_1. Now $L_{11} \cdot L_{1s} = 0$, hence $\rho(L_{11}, L_{1s}) > 0$. As R_1 is a continuous curve, there are continuous curves L_{11}^* and L_{1s}^* in R_1 containing L_{11} and L_{1s} and such that $L_{11}^* \cdot L_{1s}^* = 0$ (see Lemma C). These sets are arcwise connected, and we can draw arcs contradicting Lemma Q.

Let M_1 be M_1' plus all components L_{1i} and L_{2j} containing points of M_1', and define M_2 similarly. Then M_1 and M_2 are closed, $M_1 \cdot M_2 = 0$, and $M_1 + M_2 \supset L_1 + L_2$. By Lemma L we can draw an arc γ_1 from L_{31}' to $L_{3,k-1}'$ which has no points in M_1 or in M_2. R_1 is thus cut into two pieces, in each of which there is at least one component L_{1i} or L_{2j}; for one contains L_{11}, and the other contains that L_{2j} containing L_{32}. Thus in each piece there are less than $m_1 + m_2$ components, the number in R_1.

If one of the resulting pieces contains more than one component, we cut it up, etc. Finally each new piece of R_1 has points of only one component, and thus K_1 and K_2 are separated in R_1. We now separate K_1 and K_2 in each other piece R_i of R also. This is possible, for if K_i $(i = 1, 2)$ has points in any R_k, it also has points on J_k.

If now there are any other two of the continua K_i and K_j, $K_i \cdot K_j = 0$, each of which has points on some new J_k, we cut R further till this is no longer true, etc. This completes the proof.

11. **The homeomorphism.** Cut R into pieces of diameter $<$ some σ. We make corresponding cuts in R' as follows. The first arc γ drawn in R cuts R into the two pieces R_1 and R_2 with boundaries J_1 and J_2 say. Draw any arc γ' crossing J' in R', cutting R' into the pieces R_1' and R_2' with boundaries J_1' and J_2'. We note that $J_1'+J_2'$ is homeomorphic with J_1+J_2, with J_k' corresponding to J_k, $k=1, 2$. Say γ_1 is an arc in R_1, cutting R_1 into pieces R_{11} and R_{12} with boundaries J_{11} and J_{12}. If a_1 and b_1 are the end points of γ_1, let a_1^* and a_2^* be the corresponding points of J_1' in the above homeomorphism. Draw an arc γ_1' crossing J_1' in R_1', with end points a_1' and b_1' close to a_1^* and b_1^* respectively (Lemma P); R_1' is divided thereby into the pieces R_{11}' and R_{12}' with boundaries J_{11}' and J_{12}'. Moreover, $J_{11}'+J_{12}'+J_2'$ is homeomorphic with $J_{11}+J_{12}+J_2$, with boundaries with the same subscripts corresponding.

In general, suppose $R_{i_1i_2\cdots i_m}$ is a piece that is present after R is cut a certain amount, and say the arc $\gamma_{i_1\cdots i_m}$ divides this set into the pieces $R_{i_1\cdots i_m1}$ and $R_{i_1\cdots i_m2}$, with boundaries $J_{i_1\cdots i_m1}$ and $J_{i_1\cdots i_m2}$. If $a_{i_1\cdots i_m}$ and $b_{i_1\cdots i_m}$ are the end points of $\gamma_{i_1\cdots i_m}$, let $a_{i_1}{}^*{}_{\cdots i_m}$ and $b_{i_1}{}^*{}_{\cdots i_m}$ be the corresponding points on $J'_{i_1\cdots i_m}$ in the homeomorphism we have already. Draw an arc $\gamma'_{i_1\cdots i_m}$ crossing $J'_{i_1\cdots i_m}$, with end points $a'_{i_1\cdots i_m}$ and $b'_{i_1\cdots i_m}$ close to the above points, dividing $R'_{i_1\cdots i_m}$ into the pieces $R'_{i_1\cdots i_m1}$ and $R'_{i_1\cdots i_m2}$, with boundaries $J'_{i_1\cdots i_m1}$ and $J'_{i_1\cdots i_m2}$. The set of boundaries with primes is now homeomorphic with the set of boundaries without primes, boundaries with the same subscripts corresponding. We note that *if $R_{i_1\cdots i_m}$ and $R_{j_1\cdots j_m}$ have common points, then $R'_{i_1\cdots i_m}$ and $R'_{j_1\cdots j_m}$ have common points, and conversely.*

Having cut R into pieces of diameter $<\sigma$ and having cut R' in a corresponding fashion, we now cut each piece of R' into pieces of diameter $<\sigma/2$ and cut each piece of R in a corresponding fashion. Next we cut each resulting piece of R into pieces of diameter $<\sigma/4$, etc. Now for any $\epsilon>0$ there is an m such that

$$\delta(R_{i_1\cdots i_m}) < \epsilon, \quad \delta(R'_{i_1\cdots i_m}) < \epsilon,$$

for any m-fold subscript.

We now establish the homeomorphism between R and R'. Let p be any point of R. It lies in either R_1 or R_2 (perhaps in both), say in R_{i_1}. Then it lies in either R_{i_11} or R_{i_12} (perhaps in both), say in $R_{i_1i_2}$, etc. Thus we have a sequence of pieces

$$R \supset R_{i_1} \supset R_{i_1 i_2} \supset \cdots \supset p.$$

The corresponding pieces in R' have a single limit point:

$$R' \supset R'_{i_1} \supset R'_{i_1 i_2} \supset \cdots \supset p'.$$

This point p' we let correspond to p.

If there are different sequences of pieces in R containing p, we have different sequences in R' defining points p'. However, all these points p' are the same. For if R, R_{i_1}, $R_{i_1 i_2}$, $\cdots$, and R, R_{j_1}, $R_{j_1 j_2}$, $\cdots$, are two sequences containing p, then each piece $R_{i_1 \cdots i_m}$ has points in common with $R_{j_1 \cdots j_m}$, namely, the point p; hence, as we saw above, $R_{i_1' \cdots i_m}$ and $R_{j_1' \cdots j_m}$ have common points. Thus the corresponding sequences in R' close down on a single point. Similarly, to each point p' in R' corresponds a single point p in R.

Finally, the correspondence is continuous. For take a point p in R and an $\epsilon > 0$. Let p' be the corresponding point in R', and choose an m so that $\delta(R_{i_1' \cdots i_m}) < \epsilon$ for all m-fold subscripts. Consider all the $R_{i_1 \cdots i_m}$ with m-fold subscripts which contain p; these include all points of R in some $V_\delta(p)$. Then if $q \subset V_\delta(p)$, the corresponding point q' is in $V_\epsilon(p')$, and the continuity is established. This completes the proof of Theorem I.

12. **Proof of Theorem II.** Let I be a circle in the plane, and let S be I plus its interior. S is self-compact, connected and locally connected, and is thus a continuous curve. That $I \sim 0$ in S follows from Lemma K.†

To show that I is irreducibly ~ 0 in S, suppose that $I \sim 0$ in S', a proper closed subset of S; we can suppose that S' is a continuous curve. Let p be a point of S not in S', and let $V_\delta(p)$ have no points in S'. Let ab be a segment of a straight line passing through p with its ends on I. Let a_1b_1 and a_2b_2 be parallel segments enclosing ab, and lying at a distance δ from ab. Then in that portion of S' between a_1b_1 and a_2b_2, the (short) arcs a_1a_2 and b_1b_2 are not connected. But if C and D are those parts of S' outside a_1b_1 and a_2b_2, by Lemma L we can draw an arc joining a_1a_2 to b_1b_2 in $S' - (C + D)$, a contradiction.

Finally, that an arc crossing I in S divides S is a special (and easily proved) case of the Jordan theorem. This completes the proof.

13. **The Jordan theorem.** Let J be a simple closed curve in the plane. Let I be a circle containing J in its interior. Draw two non-intersecting line segments from I to J. $S = I$ plus its interior is thus cut into three closed 2-cells, one of which, say R, has the boundary J. Then $R - J$ is the inside of J. The points of J are obviously accessible from either side.

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† For S is a closed 2-cell.

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CONCERNING SIMPLE PLANE WEBS

BY

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A compact continuum W is said [1](¹) to be a simple web if there exists an upper semi-continuous collection G of mutually exclusive continua filling up W and another such collection H also filling up W such that (1) G is a dendron with respect to its elements and so is H and (2) if g and h are elements of G and H, respectively, the common part of g and h exists and is totally disconnected. Hence, a simple web is a web(²).

Examples of simple plane webs. Although a simple web is not necessarily a subset of the plane, this paper will deal only with those of this type. A square plus its interior is a simple web. We may consider the elements of G to be intervals parallel to one pair of sides of the square and the elements of H to be intervals parallel to the other pair of sides. We see from Theorem 1 of this paper that if $C_1, C_2, \cdots$ are circles no two of which intersect each other and such that C_1 incloses C_i $(i=2, 3, \cdots)$, then C_1 plus its interior minus the sum of the interiors of $C_2, C_3, \cdots$ is a simple web.

A square plus its interior plus an interval intersecting both the interior and the exterior of the square is not a simple web. Hence, a plane web is not necessarily a simple web. By the use of Theorem 1 of this paper, we find that the continuous curve described on page 273 of [7] which is left connected but not locally connected on the omission of some countable subset is not a simple web.

This paper gives necessary and sufficient conditions that a compact plane continuum be a simple plane web. Professor R. L. Moore suggested the problem of finding such conditions. Much credit is due him for the development of this paper.

It has been shown [6] that every simple plane web is a continuous curve. In the present paper, the following theorem will be established.

THEOREM 1. *In order that a compact plane continuous curve be a simple plane web, it is necessary and sufficient that it remain connected and locally connected on the omission of any countable subset.*

Presented to the Society, September 17, 1945; received by the editors May 12, 1945 and, in revised form, December 24, 1945.

(¹) Numbers in brackets refer to the references cited at the end of the paper.

(²) A web has been defined by Moore [3] to be a compact continuum M for which "there exist two upper semicontinuous collections H_1 and H_2 of mutually exclusive continua such that (1) each of these collections fills up M, (2) each of them is a dendron with respect to its elements, and (3) there exists an uncountable subcollection W of the collection H_1 such that no element of W is a subset of any element of H_2."

Proof of necessity. The following argument is similar to that found in the proof of Theorem 1 of [1]. Suppose that G and H are collections satisfying with respect to the simple plane web W all(3) of the conditions described in the preceding definition. Suppose that K is a countable subset of W. If W_Z $(Z=G, H)$ denotes the sum of all of the continua of the collection Z that do not intersect K, then any two points of W_G+W_H lie on a continuum which is the sum of three continua of the collection $G+H$ that do not intersect K. As W_G+W_H is dense in W, $W-K$ is connected.

If $W-K$ is not locally connected at P, there exist a domain(4) d containing P and a sequence of points of $W-K$ converging to P such that each belongs to a different component of $(W-K)\cdot d$. Let C_1, C_2, C_3 denote three circles such that C_1 incloses P, C_2 incloses C_1, C_3 incloses C_2 but no element of G or H and C_3 plus its interior is a subset of d. Now C_1 incloses a sequence of points $P_1, P_2, \cdots$ of W_G no two of which belong to the same component of $(W-K)\cdot d$ such that if g_i is the component containing P_i of the common part of d and the element of G containing P_i, then $g_1, g_2, \cdots$ has a sequential limiting set F. There exists a sequence $Q_1, Q_2, \cdots$ in W_H converging to a point of C_2 such that Q_n is a point of a subcontinuum of g_m $(m\geqq n)$ irreducible from C_1 to C_3. If h_i denotes the component containing Q_i of the common part of d and the element of H containing Q_i, then some nondegenerate subcontinuum of F belongs (see [6, Theorem 2]) to the limiting set of $h_1, h_2, \cdots$. However, as G and H are upper semi-continuous collections, this nondegenerate continuum is both a subset of an element of G and a subset of an element of H. This is contrary to the hypothesis that the common part of an element of G and an element of H is totally disconnected.

This establishes the necessity of the condition of Theorem 1. A number of preliminary theorems will be proved before the establishment of its sufficiency. When the term "simple plane web" is used in the following theorems, it is to be understood that we refer to a compact plane continuum that remains connected and locally connected on the omission of any countable subset.

THEOREM 2. *Each complementary domain of a simple plane web is bounded by a simple closed curve but no two boundaries of its individual complementary domains have more than one point in common.*

Proof. The outer boundary of a bounded complementary domain of a continuous curve is a simple closed curve [4]. As a simple plane web has no cut points, the boundary of one of its complementary domains is also the outer boundary of this domain. If the boundaries of two such complementary domains intersected in a pair of points, this pair of points would separate the simple plane web.

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(3) However, in the argument no use is made of the condition that G and H are dendrons with respect to their elements.

(4) A domain is an open set not necessarily connected. However, a complementary domain is connected [7, pp. 11 and 153].

THEOREM 3. *If the connected set x is the sum of a countable number of points and complementary domains of the simple plane web W, then each point of $W\cdot\bar{x}$ is a limit point of $W-W\cdot\bar{x}$.*

Proof. Suppose that d is a domain containing a point P of $W\cdot\bar{x}$. As $W-W\cdot x$ is a connected and locally connected inner limiting set, the common part of it and d contains [8] an arc AQB. There exists in W an arc ACB containing no point of the countable point set $W\cdot x+Q$. The point set $AQB+ACB$ contains a simple closed curve J which contains an interval $A'QB'$ of AQB. There exists in W an arc α from Q to $J-A'QB'$ which contains no point of $W\cdot x+A'+B'$. The arc α contains an arc $Q'OL$ having in common with J only its end points Q' and L which belong to $A'QB'$ and $J-A'QB'$ respectively. Let $Q'A'L$ and $Q'B'L$ be arcs lying on J with points as indicated. Then $Q'A'L+Q'B'L+Q'OL$ divides space into three mutually exclusive domains only one of which contains the connected set x. Now one of the segments(5) $\langle Q'A'L\rangle$, $\langle Q'B'L\rangle$ and $\langle Q'OL\rangle$ contains no point of $\bar{x}$ and each of them contains a point of d. Therefore d contains a point of $W-W\cdot\bar{x}$. Since any domain containing a point of $W\cdot\bar{x}$ contains a point of $W-W\cdot\bar{x}$, then each point of $W\cdot\bar{x}$ is a limit point of $W-W\cdot\bar{x}$.

THEOREM 4. *Under the hypotheses of Theorem 3, the closure of x does not separate W.*

Proof. If P is a point of the plane S, then $S-(x+P)$ is a connected and locally connected inner limiting set. Hence [8] it is arcwise connected. Furthermore [2, Theorem 12], $S-\bar{x}$ is connected.

Suppose that $W-W\cdot\bar{x}$ is the sum of two mutually separated sets S_1 and S_2. There exists in $S-\bar{x}$ an arc AB from a point A of S_1 to a point B of S_2. Let P denote the last point of the closed point set $S_1\cdot AB$ in the order from A to B on AB. There is a point Q following P on AB such that the interval PQ of AB contains no point of S_2. Let P' denote the first point in the order from Q to B that the interval QB of AB has in common with β, the boundary of d, the complementary domain of W that contains Q. Since the closure of $x+d$ does not separate S, one arc of β from P to P' contains no point of $\bar{x}$. But P belongs to S_1 and P' belongs to S_2. Thus, the supposition that $W-W\cdot\bar{x}$ is not connected leads to a contradiction.

THEOREM 5. *If R is a connected point set consisting of a plane domain plus a totally disconnected point set and the complement of R is a locally connected inner limiting set, then $\bar{R}$ is a continuous curve.*

Proof. If $\bar{R}$ is not locally connected, there exist a domain d, a sequence of components $c_1, c_2, \cdots$ of $\bar{d}\cdot\overline{R}$, and a sequence of arcs $\alpha_1, \alpha_2, \cdots$ such that (1) for each positive integer j, α_j is a subset of $\bar{d}-\bar{d}\cdot\overline{R}$; (2) if h, i, j are positive integers such that $h<i<j$, then α_i separates c_h from c_j in d and c_i separates α_h from α_j in d; (3) d intersects c, the sequential limiting set of $c_1, c_2, \cdots$. The sequence $\alpha_1, \alpha_2, \cdots$ converges to c.

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(5) By $\langle\alpha\rangle$ is meant the arc α minus its end points.

As no point of c is a point of a domain lying in R and some nondegenerate subcontinuum c' of c is a subset of d, the complement of R contains two points P_1 and P_2 of c'. There exist domains d_1 and d_2 containing P_1 and P_2 respectively such that d_1 and d_2 are mutually exclusive subsets of d. As the complement of R is locally arcwise connected at P_1 and P_2, there exists a positive integer j such that each of the point sets $d_1-d_1\cdot R$ and $d_2-d_2\cdot R$ contains an arc intersecting both α_j and α_{j+2}. Therefore, there is a simple closed curve J lying in $d-d\cdot R$ and intersecting α_j and α_{j+2}. As R is connected and contains no point of J, it does not intersect both the interior and the exterior of J. Therefore, no component of $\bar{d}\cdot\overline{R}$ separates α_j from α_{j+2} in d and $\overline{R}$ is locally connected.

THEOREM 6. *Under the hypotheses of Theorem* 3, $\bar{x}$ *is a continuous curve each of whose degenerate cyclic elements is a point of* W *in* x *and each of whose nondegenerate cyclic elements is the closure of a complementary domain of* W *in* x.

Proof. As the complement of x is a locally connected inner limiting set, it follows from Theorem 5 that $\bar{x}$ is a continuous curve. If P is a cut point of $\bar{x}$, it is a cut point of x and therefore must be a point of W in x.

If J is a simple closed curve in $\bar{x}$, we shall show that it is a subset of the closure of a complementary domain of W in x. Since $\bar{x}$ does not separate space (see Theorem 4), one of the complementary domains of J is a subset of $\bar{x}$. By Theorem 3, we have that J separates no two points of $\bar{x}-x$ from each other. Therefore, one of the complementary domains of J is a subset of x and is therefore a subset of a complementary domain of W in x.

THEOREM 7. *Suppose* W *is a simple plane web,* β *is its outer boundary and* X *is the collection of all nondegenerate continua* x *such that* x *is maximal with respect to being the closure of a connected set which is the sum of a countable number of points and bounded complementary domains of* W. *If* A *and* B *are points of* W, *there is an arc* AB *in* W *such that* AB *minus its end points contains no point of either* β *or any element of* X.

Proof. Suppose that A belongs to x_A of X and that B belongs to neither β nor any element of X. As $W-W\cdot x_A+A$ is connected, A is accessible [10, Theorem 1] from $W-W\cdot x_A$. Since the continua of the collection X are mutually exclusive and no one of them separates the plane, then x_A plus the complement of the sum of the elements of X is a connected and locally connected inner limiting set which if regarded as space satisfies Axioms 1–5 of [7]. In this space, there exists an arc AB having no point except A in common with

$x_A+\beta$. The arc AB is a subset of W.

THEOREM 8. *Under the hypotheses of Theorem 7, if P is a point belonging to both β and an element x_P of X, Q is another point of β and c is a component of x_P-P, then there exists in W a simple closed curve J inclosing c but no point of x_P-c and containing P and Q but no other point either of β or of an element of X.*

Proof. Let X' denote the set of all continua of the collection X except x_P and x_Q where x_Q denotes the element of X containing Q if there is one but denotes Q if there is not. Let Z denote the collection whose elements are the continua of X' and the points of the plane that belong to no element of X'. The collection Z is upper semi-continuous and no one of its elements separates the plane. It follows [5] that the space Σ_1 whose "points" are elements of Z is topologically equivalent to the plane. If Σ_2 denotes the space whose "points" are the "points" of Σ_1 other than the elements of X' not intersecting β, then it is topologically equivalent to a plane minus a countable number of points. If β' denotes the "point" set whose points are the points of β that belong to no element of X' and the continua of X' that intersect β, then in the space Σ_2, β' is a simple closed curve and x_P and x_Q are continuous curves each lying except for one point wholly in the interior of β'. In this space, there exists an arc EQ from a point E of c to Q such that $x_P+x_Q+\beta'$ intersects EQ only at $E+Q$. Furthermore [7, Theorem 12, p. 217], there exists a simple closed curve intersecting $x_P+x_Q+\beta'$ only at $P+Q$ and inclosing $c+EQ-Q$ but no point of x_P-c.

Description of dendrons G and F. In this section, W will designate a compact plane continuous curve which is left connected and locally connected on the omission of any countable point set. Also, X will denote the set of all nondegenerate continua x such that x is maximal with respect to being the closure of a connected set which is the sum of a countable number of points and bounded complementary domains of W.

As a step toward proving the sufficiency of Theorem 1, we shall describe an upper semi-continuous collection G of mutually exclusive continua filling up W such that G is a dendron with respect to its elements. At the same time, we shall describe a dendron F in W such that each element of G intersects F in only one point.

As the description of G is somewhat complicated, we shall describe it by steps. First, we consider a transformation T throwing a square plus a subset of its interior into W. Next, we describe an upper semi-continuous collection of mutually exclusive continua which is an arc with respect to its elements and which covers a part of the inverse of W under transformation T. We cover more of this inverse by another upper semi-continuous collection of mutually exclusive continua. The process is continued until we have an upper semi-continuous collection of mutually exclusive continua filling up the inverse of W under T such that this collection is a dendron with respect to its elements. The image of the elements of this collection with respect to T gives the elements of G.

Transformation T. Let $P_1P_2P_3$ and $P_1P_4P_3$ be two arcs whose sum is the boundary β of the unbounded complementary domain of W and whose ends are not points of X^*(⁶). There exist in W arcs $P_1P_5P_3$ and $P_1P_6P_3$ which except for their end points are mutually exclusive and contain no point of $X^*+\beta$. Let R be the interior of a square with diagonals V_1V_3 and V_2V_4 and let V_5, V_6 be the midpoints of V_1V_4, V_2V_3 respectively.

There exists a continuous transformation T of $\overline{R}$ into $W+X^*$ such that $T(V_1V_5) = T(V_4V_5) = P_1P_5$, $T(V_2V_6) = T(V_3V_6) = P_3P_6$, $T(V_6V_2) = P_6P_3$, $T(V_4V_5) = P_1P_6$ and each point of $W+X^*-P_1P_5-P_6P_3$ is the image of only one point of $\overline{R}-V_1V_5V_4-V_2V_6V_3$ under the transformation T.

Let M denote the set of all points P of $\overline{R}$ such that the image of P under T is a point of W. We note that M is a compact continuous curve which is left connected and locally connected on the omission of any countable subset. Let Y be the collection of all continua y such that y is the inverse under T of an element of X. We note that y is maximal with respect to being the closure of a connected set which is the sum of a countable number of points and bounded complementary domains of M.

The elements of Y are ordered $y_1, y_2, \cdots$. By Theorem 7 and the fact that M is separable, there exists a countable point set $z_1, z_2, \cdots$ dense in M and such that for each positive integer i, z_i belongs to M but not to Y^* or the boundary of R. The bounded complementary domains of M are ordered $d_1, d_2, \cdots$.

We shall describe an upper semi-continuous collection G' of mutually exclusive continua filling up M such that (1) G' is a dendron with respect to its elements, (2) no element of G' contains more than two points of Y^*, (3) each element of G' intersects V_6V_2 and V_4V_5, and (4) each element of G' is a continuum of condensation of M. Also, we shall describe a dendron F' in M such that each element of G' intersects F' in only one point.

First step. If some element of Y has a point P in common with V_1V_2 or V_4V_3 and P is nearer the midpoint than either end point of the one of V_1V_2, V_4V_3 on which it lies, let i be the least positive integer n such that y_n is such an element of Y and let P be the point that y_i has in common with $V_1V_2+V_4V_3$. If there is no such i, let P be the midpoint of V_1V_2. Let P_1 and P_2 be the projections of P on V_1V_2 and V_4V_3 respectively. Let k be the least positive integer n, if there is one, such that y_n has no point in common with the boundary of R. There are two arcs P_1AP_2 and P_1DP_2 lying in M, intersecting Y^* plus the boundary of R only at P_1 and P_2 and forming a simple closed curve J such that (1) J incloses z_1 and y_k, (2) the interior of J together with P_1 and P_2 contains each element of Y (if any) that contains

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(⁶) By X^* is meant the set of all points P such that P is a point of an element of X.

P_1 or P_2, and (3) P_1AP_2 separates $\langle P_1DP_2\rangle$ from V_1V_4 in M.

We shall consider the case in which P_1 and P_2 belong to elements $y(P_1)$ and $y(P_2)$ of Y respectively. Let $\langle P_1BP_2\rangle$ and $\langle P_1CP_2\rangle$ be mutually exclusive open arcs in M which are inclosed by J, which contain no point of Y^*, and which are such that P_1BP_2 separates $\langle P_1CP_2\rangle$ from V_1V_4 in M, the simple closed curve formed by $P_1AP_2+P_1BP_2$ incloses $y(P_1)-P_1$, the simple closed curve formed by $P_1BP_2+P_1CP_2$ incloses $y(P_2)-P_2$, and P_2C of P_1CP_2 intersects V_6V_2.

If $y(P_1)-P_1$ is not connected, let $c_1, c_2, \cdots$ denote its components. Let i, j be the least positive integers m, n respectively such that z_m, y_n are inclosed by $P_1AP_2+P_1BP_2$. There is a simple closed curve J_1 in M containing P_1 and P_2 and inclosing c_1, z_i, y_j but no point of $y(P_1)-c_1$ which is such that $J_1-P_1-P_2$ contains no point of Y^* and is inclosed by $P_1AP_2+P_1BP_2$. Let O represent the simple closed curve which is a subset of $P_1AP_2+P_1BP_2+J_1$ and which incloses c_2 but no point of J_1. Let i, j be the least positive integers m, n such that z_m, y_n are inclosed by O. There is a simple closed curve J_2 in M containing P_1 and P_2 and inclosing c_2, z_i, y_j but no point of $Y(P_1)-c_2$ which is such that $J_2-P_1-P_2$ contains no point of Y^* and is inclosed by O. This process is continued until for each c_i there is a unique J_i inclosing it.

The interior of $P_1AP_2+P_1BP_2$ is topologically equivalent to the bounded complementary domain of $P_1BP_2+P_2C+P_1DP_2$. Treating $P_2BP_1+P_1DP_2$ as P_1AP_2 and P_2C+CP_2 as P_1BP_2, we consider curves $J_1', J_2', \cdots$ inclosing the components of $y(P_2)-P_2$ in a manner similar to that in which we considered simple closed curves inclosing the components of $y(P_1)-P_1$.

Let g_1' denote the closure of $J+P_1BP_2+P_2C+J_1+J_2+\cdots+J_1'+J_2'+\cdots$. The continuum g_1' is a subset of an element of G'. It is to be noted that g_1' is a continuum of condensation of M intersecting both V_5V_2 and V_4V_6 and if C is the common part of M and a bounded complementary domain of g_1', then (1) C intersects the interiors of triangles $V_1V_2V_5$ and $V_4V_3V_6$, (2) $\overline{C}$ is a simple plane web, and (3) no two elements of Y intersect both C and the boundary of C with respect to M.

Second step. Let i be the least positive integer n such that y_n intersects one of the intervals V_1P_1 and V_4P_2 in a point P nearer the midpoint than either end point of that interval. If there is no such i, let P be the midpoint of V_1P_1. Let P_3, P_4 be the projections of P on V_1P_1 and V_4P_2 respectively. By treating $V_1P_1+P_1AP_2+P_2V_4+V_4V_1$ as the boundary of R and following a procedure similar to that used in finding g_1', we obtain g_2'. By treating $P_1V_2+V_2V_3+V_3P_2+P_2DP_1$ as the boundary of R, we get g_3'. This process is continued until every point of M belongs to either the closure of the sum of all the g_n''s or to a bounded complementary domain of one of them.

If U is a component of M minus the common part of M and the sum of the g_n''s together with their bounded complementary domains and U is closed, U will have a point in common with V_1V_2 and a point in common with V_4V_3

and is to be considered as a g'. If U is not closed, it will not have a point in common with either V_1V_2 or V_4V_3 and, as every two g_n''s are separated in M by a third g_n', there is only one g_n' which contains a limit point of U. If there are such U's, g_n' plus all U's which have a point of g_n' as a limit point is a g'. Each g' described is a subset of an element of G'.

If G_1' is the set of maximal g''s thus described, G_1' is an arc with respect to its elements, each point of $G_1'^*$ is a limit point of $M'-G_1'^*$, no element of G_1' has two points in common with an element of Y and each element of G_1' has both a point in common with V_1V_2 and a point in common with V_4V_3. There exists a dendron F_1' in M such that $F_1' \cdot G_1'^*$ is V_4V_3, $F_1' - V_4V_3$ intersects neither V_4V_6 nor any element of Y and if C is a component of $M-G_1'^*$, then $F_1' \cdot C$ is an arc minus an end point of diameter less than the diameter of V_1V_2 and if K is a connected subset of C that intersects both this arc and an element of Y that intersects the boundary of C with respect to M, then K intersects both V_5V_2 and V_4V_6. Let D_1 be a domain containing $F_1' - V_4V_3$ such that each point of D_1 is nearer F_1' than $G_1'^* + V_4V_6$. Then $\overline{D}_1$ intersects $G_1'^*$ only in V_4V_3.

Third step. If C is a component of $M-G_1'^*$, C is the common part of M and a bounded complementary domain of some element of G_1'. We now consider the breaking up of $\overline{C}$ into elements of G'. We note that C intersects the interiors of the triangles $V_1V_2V_5$ and $V_4V_3V_6$. If β denotes the boundary of C with respect to M, then $C+\beta$ is a simple plane web and no more than one element of Y intersects both C and β.

Let y_a be such an element of Y intersecting β at P_1 and let c denote the closure of the common part of y_a and C. Let Q_1Q_2 be an arc such that $Q_1Q_2-Q_1$ is $F_1' \cdot C$. Let i be the least positive integer n such that d_n is a subset of c. There is a point P_2 of the boundary of d_i such that $c+\beta-P_2$ is connected. There is a subcontinuum w of c such that (1) w contains P_1 and P_2, (2) if a point of a complementary domain of M belongs to w, the complementary domain is a subset of w, and (3) no subcontinuum of w has properties (1) and (2). There is an upper semi-continuous collection E of mutually exclusive continua filling up w such that E is an arc from P_1 to P_2 with respect to its elements and such that every element of E is either a point or an arc which has only its end points in common with the boundary of c. Designate the arc of elements of E by $\xi(P_1P_2)$. Each connected subset of C that intersects $\xi(P_1P_2)$ and Q_1Q_2 also intersects V_5V_2 and V_4V_6. To adjust the argument to the case where no element of Y has a point in common with both C and β, let $\xi(P_1P_2)$ be an arc in $C+P_1$ from a point P_1 of β to a point P_2 of C such that $\xi(P_1P_2)$ contains no point of Y^* and any connected subset of C intersecting both $\xi(P_1P_2)$ and Q_1Q_2 also intersects both V_5V_2 and V_4V_6.

Let j, k be the least positive integers m, n respectively such that z_m, y_n are subsets of C. There is a simple closed curve J lying in C, inclosing y_k, intersecting $\xi(P_1P_2)+Q_1Q_2$ only at P_2+Q_2 and such that Q_1Q_2+J contains z_j.

We note that J plus the common part of M and the interior of J is a simple plane web intersecting both V_5V_2 and V_4V_6.

A breaking up of the portion of $\overline{C}$ not in the interior of J into elements of G' will now be considered. Considering the elements of $\xi(P_1P_2)$ as points, $\xi(P_1P_2)$ as V_1V_2, β as V_1V_4, $\xi(P_1P_2)$ as V_4V_3, J as V_2V_3, and Q_1Q_2 as V_5V_6, we divide the portion of $\overline{C}$ not in the interior of J in a manner somewhat analogous to that described in the second step. We divide the portion of $\overline{C}$ corresponding to $V_1V_2V_6V_6$ in the manner in which $V_1V_2V_3V_4$ was divided in the second step and we divide the portion of $\overline{C}$ corresponding to $V_6V_6V_3V_4$ in the same fashion, care being taken that each pair of g''s thus defined that intersect on Q_1Q_2 also intersect the same elements of $\xi(P_1P_2)$.

The closures of all components of $M-G_1'^*$ are broken up just as was $\overline{C}$. If a g' as defined in this step intersects another g' defined in either this or the second step, the sum of the two is a g'. Let G_2' denote the set of maximal g''s defined. We note that the boundary of d_i of y_a is a subset of $G_2'^*$. Also, each element of G_2' intersects V_6V_2 and V_4V_6 and so does each component of $M-G_2'^*$.

There exists a dendron F_2' in M such that $F_2' \cdot G_2'^*$ is F_1', $F_2'-F_1'$ is a subset of D_1, $F_2'-F_1'$ intersects no element of Y, and if C is a component of $M-G_2'^*$, then $F_2' \cdot C$ is an arc minus one end point of diameter less than one-half the diameter of V_1V_2. Let D_2 be a domain containing $F_2'-F_1'$ such that each point of D_2 is nearer F_2' than $G_2'^* \overline{D}_2$ and is a subset of D_1+F_1'.

Fourth step. We shall now describe the breaking up of a component C of $M-G_2'^*$. The boundary of C with respect to M is a simple closed curve J. If an element of Y intersects both C and J, break up $\overline{C}$ as described in the third step. Otherwise, let j be the least integer equal to n such that y_n is inclosed by J. Let P be a point of y_j such that y_j-P is connected. Let $Q_1Q_2-Q_1$ be the common part of F_2' and the interior of J. There is an arc Q_1P in M which except for its end points intersects neither $J+Q_1Q_2$ nor any element of Y and which is such that any connected subset of the interior of J intersecting Q_1Q_2 and y_j but not Q_1P intersects both V_5V_2 and V_4V_6. Treating Q_1P+J, Q_1Q_2, and y_j as β, Q_1Q_2, and c respectively were treated in the third step, we break up the common part of M and the interior of J in a manner analogous to that in which C was broken up as considered in the third step.

Let G_3' be the set of all g''s considered after all components of $M-G_2'^*$ are broken up as described above. We define a dendron F_3' in D_2 containing F_2' and satisfying conditions analogous to those satisfied by F_2'. In like manner, we consider G_4', G_6', $\cdots$ and F_4', F_5', $\cdots$. It is to be noted that (1) for each positive integer j, there is an n such that z_j and the boundary of d_i belongs to $G_n'^*$, (2) for no n do three points of Y^* belong to the same element of G_n', (3) each element of G_n' intersects V_6V_2 and V_4V_6, and (4) each component of $M-G_n'^*$ intersects V_5V_2 and V_4V_6.

Definition of dendron G. Let G' be the set of all continua such that g'

is an element of G' only if either (1) for some positive integer n, g' belongs to G_n' and G_{n+1}' or (2) g' is a component of $M-(G_1'^*+G_2'^*+\cdots)$. With respect to its elements, G' is a dendron. Each element of G' is a continuum intersecting both V_6V_2 and V_4V_6. Each element of G' is a continuum of condensation of M. No element of G' intersects an element of Y in more than two points.

We define G to be the collection of all point sets g such that g is the image of an element of G' under T. We note that G is an upper semi-continuous collection of mutually exclusive continua filling up W such that G is a dendron with respect to its elements.

Definition of dendron F. We define F' to be the limiting set of F_1', F_2', $\cdots$. We note that it is a dendron that intersects each element of G' in only one point. We define F to be the image of F' under transformation T. We note that F is a dendron that intersects each element of G in only one point and the outer boundary of W in only an arc.

Description of arc H. We shall describe an upper semi-continuous collection H of mutually exclusive continua filling up W such that H is an arc with respect to its elements and such that if g and h are elements of G and H respectively, then $g\cdot h$ is totally disconnected. First, we shall establish four additional theorems.

THEOREM 9. *If R is the interior of a square and E is a collection of point sets covering $\overline{R}$ such that for each point P and each domain D there is a point P' of D which belongs to no element of E that contains P, then for any positive ϵ and any two points A and B of $\overline{R}$ there is in $R+A+B$ an arc ACB such that* (1) *for any two points Q_1 and Q_2 of ACB*(7), $\cos(Q_1Q_2, AB)>1-\epsilon$ *and* (2) *no element of E contains a subarc of ACB.*

Proof. There is a point P_1 such that(8) $d(A, P_1)=d(P_1, B)$, the intervals AP_1 and P_1B are subsets of $R+A+B$, and $\cos(AP_1, AB)>1-\epsilon$. There exists a quadrilateral Q lying in $R+A+B$ and inclosing $AP_1+P_1B-(A+B)$. Within Q there exist (1) a point P_2 not belonging to any element of E that contains A and such that $\max[d(A, P_2); d(P_2, P_1)]<(2/3)d(A, P_1)$, $\cos(AP_2, AB)>1-\epsilon$ and $\cos(P_2P_1, AB)>1-\epsilon$ and (2) a point P_3 not belonging to any element of E that contains P_1 and such that $\max[d(P_1, P_3); d(P_3, B)]<(2/3)d(P_1, B)$, $\cos(P_1P_3, AB)>1-\epsilon$ and $\cos(P_3B, AB)>1-\epsilon$. Likewise we get within Q a point P_4 not belonging to any element of E containing A and such that $\max[d(A, P_4); d(P_4, P_2)]<(2/3)d(A, P_2)$ and $\cos(AP_4, AB)>1-\epsilon$ and $\cos(P_4P_2, AB)>1-\epsilon$. This process is continued.

Let L denote the closure of the sum of all such P_n's. Since L is closed and compact and as its projection on the line AB is the interval AB and each line perpendicular to this interval contains only one point of L, then L is an arc from A to B.

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(7) By $\cos(AB, CD)$ is meant the cosine of the smaller angle between the lines AB and CD. If AB and CD are coincident or parallel, $\cos(AB, CD)=1$.

(8) By $d(A, B)$ is meant the distance between A and B.

As each of the numbers $\cos(P_2P_1, AB)$ and $\cos(P_1P_3, AB)$ is greater than $1-\epsilon$ and as the projection of P_1 on AB is between the projections of P_2 and P_3, it follows that $\cos(P_2P_3, AB) > 1-\epsilon$. By induction, we get for different integers j and k that $\cos(P_jP_k, AB) > 1-\epsilon$. Hence if Q_1 and Q_2 are points of the continuum L, $\cos(Q_1Q_2, AB) > 1-\epsilon$.

Each subarc of L contains a P_j and a P_k such that P_j does not belong to any element of E containing P_k. Hence, no subarc of L is a subset of an element of E.

THEOREM 10. *Let R be the interior of a square with diagonals V_1V_3 and V_2V_4; let E be a collection of mutually exclusive point sets filling up $\overline{R}$ such that* (1) *no element of E contains a nondegenerate connected subset of $V_1V_4+V_2V_3$ and* (2) *no element of E is dense in a domain; then there exists an upper semi-continuous collection H of mutually exclusive arcs filling up $\overline{R}$ such that H is an arc with respect to its elements, each element of H has one end point on V_1V_2 and one on V_4V_3, each point of $V_1V_2+V_4V_3$ is an end point of an element of H, and no element of E contains a subarc of an element of H.*

Proof. With the help of the preceding theorem, it will be shown that there exists a set of mutually exclusive arcs $\alpha_1, \alpha_2, \cdots$ such that (1) each of them has one end point on V_1V_2 and the other on V_4V_3 and has no arc in common with an element of E and (2) the components of $\overline{R}-(\alpha_1+\alpha_2+\cdots)$ are arcs having one end point on V_1V_2 and one on V_4V_3 and, furthermore, no such component has an arc in common with an element of E.

We shall assume that $\overline{R}-R$ is a unit square. Consider points Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 in the orders $V_1Q_1Q_2Q_3V_4$ and $V_2Q_4Q_5Q_6V_3$ and such that Q_1, Q_2, Q_3 and Q_4, Q_6, Q_6 divide V_1V_4 and V_2V_3 respectively into four equal parts. Choose a positive number ϵ less than 1. Let P_0 be the midpoint of V_1V_2 and P_1, P_2 be points of V_1V_2 at a distance from P_0 equal to $\epsilon/6$ and in the order $V_1P_1P_0P_2V_2$. Let P_3 and P_4 denote points on V_1V_2 at a distance from P_0 equal to $2\epsilon/6$ and in the order $V_1P_3P_0P_4V_2$. Similarly, consider points $P_5, P_6, \cdots, P_{2j-1}, P_{2j}$ where P_{2j-1} and P_{2j} are at a distance from P_0 equal to $j\epsilon/6$ and from V_1 and V_2 respectively of not more than $\epsilon/6$ and are in the order $V_1P_{2j-1}P_0P_{2j}V_2$. Let P_{2j+1} be the midpoint of V_1P_{2j-1} and in general let $P_{2j+2k+1}$ be the midpoint of $V_1P_{2j+2k-1}$ for all positive integers k. Also $P_{2j+2k+2}$ is the midpoint of $P_{2j+2k}V_2$. Let $P_{i,1}$, $P_{i,2}$, $P_{i,3}$, $P_{i,4}$ be the projections of P_i on Q_1Q_4, Q_2Q_5, Q_3Q_6, V_4V_3 respectively.

It is assumed that each of the following arcs has the properties that (1) no subarc of it is a subset of E and (2) if A and B are two points of it, $\cos(AB, V_1V_4) > 1/(1+\epsilon^2)^{1/2}$. There exist in P_0+P_2+R mutually exclusive arcs $P_0P_{0,1}$ and $P_2P_{2,1}$ satisfying the above two conditions and such that no arc from P_0P_2 to $P_{0,1}P_{2,1}$ in $P_0P_2+P_{0,1}P_{2,1}$ plus the interior of $P_0P_2+P_2P_{2,1}$

$+P_{2,1}P_{0,1}+P_{0,1}P_0$ is a subset of an element of E. In the case where the elements of E contain arcs, this may be seen by constructing $P_0P_{0,1}$ and $P_2P_{2,1}$ so that two elements of E contain arcs from $P_0P_{0,1}$ to $P_2P_{2,1}$ in $P_0P_{0,1}+P_2P_{2,1}$ plus the interior of $P_0P_2+P_2P_{2,1}+P_{2,1}P_{0,1}+P_{0,1}P_0$. Also, there exist in R mutually exclusive arcs $P_{0,2}P_{0,3}$ and $P_{2,2}P_{2,3}$ which satisfy the above two conditions and such that no arc from $P_{0,2}P_{2,2}$ to $P_{0,3}P_{2,3}$ in $P_{0,2}P_{2,2}+P_{0,3}P_{2,3}$ plus the interior of $P_{0,2}P_{2,2}+P_{2,2}P_{2,3}+P_{2,3}P_{0,3}+P_{0,3}P_{0,2}$ is a subset of an element of E. Furthermore, there are mutually exclusive arcs $P_{0,1}P_{0,2}$ and $P_{1,1}P_{1,2}$ in R satisfying the above two conditions and such that no arc from $P_{1,1}P_{0,1}$ to $P_{1,2}P_{0,2}$ in $P_{1,1}P_{0,1}+P_{1,2}P_{0,2}$ plus the interior of $P_{1,1}P_{0,1}+P_{0,1}P_{0,2}+P_{0,2}P_{1,2}+P_{1,2}P_{1,1}$ is a subset of an element of E. Arcs satisfying corresponding conditions join $P_{1,3}$, $P_{0,3}$ to $P_{1,4}$, $P_{0,4}$ respectively. Furthermore, arcs which intersect no arc previously described and which satisfy conditions analogous to those given join P_3, P_1 to $P_{3,1}$, $P_{1,1}$; $P_{3,2}$, $P_{1,2}$ to $P_{3,3}$, $P_{1,3}$; $P_{2,1}$, $P_{4,1}$ to $P_{2,2}$, $P_{4,2}$; $P_{2,3}$, $P_{4,3}$ to $P_{2,4}$, $P_{4,4}$; $P_{5,1}$, $P_{3,1}$ to $P_{5,2}$, $P_{3,2}$; $\cdots$ in such a manner that if α_i designates $P_iP_{i,1}+P_{i,1}P_{i,2}+P_{i,2}P_{i,3}+P_{i,3}P_{i,4}$, then V_1V_4, V_2V_3 are the sequential limiting sets of $\alpha_1, \alpha_3, \cdots$ and $\alpha_0, \alpha_2, \cdots$ respectively.

Let H_1 be the set consisting of V_1V_4, V_2V_3, $\alpha_0, \alpha_1, \cdots$. If h_1 and h_2 are elements of H_1 which are not separated from each other in $\overline{R}$ by any element of H_1 and C is set of all points of $\overline{R}$ between h_1 and h_2, no arc from Q_2Q_5 to $V_1V_2+V_4V_3$ in $Q_2Q_5+V_1V_2+V_4V_3$ plus the bounded complementary domain of $V_1V_2+V_4V_3+h_1+h_2$ is a subset of an element of E.

Designate $V_1V_2\cdot\overline{C}$ and $V_4V_3\cdot\overline{C}$ by U_1U_2 and U_4U_3 respectively, where the points U_1, U_2, U_3, U_4 are in the orders $V_1U_1U_2V_2$ and $V_4U_4U_3V_3$. Treating U_1U_2 and U_4U_3 as V_1V_2 and V_4V_3 were treated, we break up $\overline{C}$ in a manner somewhat analogous to that in which $\overline{R}$ was divided except that instead of three lines Q_1Q_4, Q_2Q_5, Q_3Q_6, we use seven lines parallel to U_1U_2 that divide a segment from U_1U_2 to U_4U_3 into eight equal parts. The division of all such C's gives us H_2. This division of C is possible, for no use was made above of the fact that V_1V_4 and V_2V_3 were straight lines. In fact, it can be shown that if (1) P_1P_2 and $P_1'P_2'$ are parallel intervals, (2) P_1P_1' and P_2P_2' are mutually exclusive arcs such that if A and B are either two points of P_1P_1' or two points of P_2P_2', then $\sin (AB, P_1P_2)>1/(1+\epsilon^2)^{1/2}$, (3) Q_1, Q_2, Q_1', Q_2' are points in the orders $P_1Q_1Q_2P_2$ and $P_1'Q_1'Q_2'P_2'$ and $\min [\sin (Q_1Q_2', P_1P_2); \sin (Q_2Q_1', P_1P_2)]>1/(1+\epsilon^2)^{1/2}$, and (4) E is a collection of mutually exclusive point sets filling up $P_1P_2+P_2P_2'+P_2'P_1'+P_1'P_1$ plus its interior and such that no element of E is dense in a domain, then there exist in its interior plus the points Q_1, Q_2, Q_1' and Q_2' two mutually exclusive arcs Q_1Q_1' and Q_2Q_2' such that no arc from Q_1Q_2 to $Q_1'Q_2'$ in $Q_1Q_2+Q_2Q_2'+Q_2'Q_1'+Q_1'Q_1$ plus its interior is a subset of an element of E and such that if C and D are two points of Q_1Q_1' or two points of Q_2Q_2', then $\sin (CD, P_1P_2)>1/(1+\epsilon^2)^{1/2}$.

No element of E contains an arc in $\overline{R}-(H_1^*+H_2^*)$ whose projection of V_1V_4 is of length 3/8. With the use of fifteen lines parallel to V_1V_2, H_3 is defined in such a way that no element of E contains an arc in $\overline{R}-(H_1^*+H_2^*+H_3^*)$ whose projection on V_1V_4 is of length 3/16. If a suitable pattern is used in defining $H_4, H_5, \cdots$ and H is the set such that h is an element of H only if it is an element of some H_n or a component of $\overline{R}-(H_1^*+H_2^*+\cdots)$, then H satisfies the conditions of our theorem.

Our method actually gives the following result.

THEOREM 11. *Let R be the interior of a square with diagonals V_1V_3 and V_4V_2; let E be a collection of point sets covering $\overline{R}$ such that no element of E contains a nondegenerate connected subset of $V_1V_4+V_2V_3$ and if D_1 and D_2 are two domains, there exist nondegenerate connected subsets f_1 and f_2 of D_1 and D_2 respectively such that no element of E intersects both f_1 and f_2. Then there exists an upper semi-continuous collection H of mutually exclusive arcs filling up $\overline{R}$ such that H is an arc with respect to its elements, each element of H has one end point on V_1V_2 and one on V_4V_3, each point of $V_1V_2+V_4V_3$ is an end point of an element of H and no element of E contains a subarc of an element of H.*

THEOREM 12. *Let R be the interior of a square with diagonals V_1V_3 and V_2V_4; let F be a dendron in $\overline{R}$ intersecting the boundary of R only in V_4V_3; let E be a collection of point sets covering $\overline{R}$ such that no element of E contains a nondegenerate connected subset of $V_1V_4+V_2V_3$ and if D_1 and D_2 are two domains, there exist nondegenerate connected subsets f_1 and f_2 in D_1 and D_2 respectively such that no element of E intersects both f_1 and f_2. There exists an upper semi-continuous collection H of mutually exclusive continua filling up $\overline{R}$ such that H is a dendron with respect to its elements, V_1V_4 is a subset of an element of H and so is V_2V_3, no element of E contains a nondegenerate subcontinuum of an element of H and each element of H intersects F in only one point and V_1V_2 in only a totally disconnected point set.*

Indication of proof. We construct dendron H so that each element of H intersects F in only one point in a manner analogous to that in which we constructed G' so that it intersected F' in only one point. We use the methods of Theorem 10 to insure that no element of H intersects an element of E in a nondegenerate continuum.

Definition of arc H. We shall now define an upper semi-continuous collection H of mutually exclusive continua filling up W such that H is an arc with respect to its elements, each element of H intersects each element of G, and the intersection of an element of G and an element of H is a totally disconnected point set.

As the points of $M-M\cdot Y^*$ and the elements of Y form an upper semi-continuous collection of mutually exclusive continua filling up $\overline{R}$, it follows [5] that there is a continuous transformation T' carrying $M+Y^*$ into $\overline{R}$ such that (1) T' carries each point of $V_5V_2+V_4V_6$ plus the portion of the boundary of R not belonging to Y^* into itself, (2) T' carries each point of $M-M\cdot Y^*$

into a point, (3) T' carries each element of Y into a point, and (4) each point of $\overline{R}$ is either the image of only one point of $M-M\cdot Y^*$ or the image of only one element of Y. If G'' is the set of all $T'(g')$'s where g' is an element of G', the elements of G'' intersect V_5V_2 and V_4V_6. By the methods of Theorem 11, it can be shown that there is an upper semi-continuous collection H'' of mutually exclusive continua filling up $\overline{R}$ such that (1) H'' is with respect to its elements an arc with ends at V_1 and V_3, (2) every element of H'' other than V_1 and V_2 is an arc having only one point on $V_1V_2+V_2V_3$ and only one point on $V_1V_4+V_4V_3$, these points being its end points, (3) if g'' is an element of G'' intersecting an element h'' of H'', $g''\cdot h''$ is totally disconnected, and (4) one element of H'' contains V_5+V_2 and another contains V_4+V_6 and each of these elements lies except for its end points on the interior of the parallelogram $V_5V_2V_6V_4$.

If H' is a set such that h' is an element of H' only if it is a subcontinuum of M which is maximal with respect to $T'(h')$ being an element of H'', then H' is a collection of mutually exclusive continua filling up M. Furthermore, it is an arc with respect to its elements. As no element of G' has more than two points in common with the boundary of an element of Y, if g' is an element of G' having a point in common with an element h' of H', $g'\cdot h'$ is totally disconnected.

Let H be a set such that h is an element of H only if it is a continuum which is maximal with respect to being the image under T of the sum of two or three elements of H'. Then H is the required collection of mutually exclusive continua filling up W. The sufficiency of Theorem 1 is demonstrated.

Theorem 1 may also be stated as follows:

THEOREM 1. *A necessary and sufficient condition that a compact plane continuum W remain connected and locally connected on the omission of any countable subset is that there exist two continuous monotone transformations T_1 and T_2 such that both $T_1(W)$ and $T_2(W)$ are nondegenerate dendrons and if a and b are elements of $T_1(W)$ and $T_2(W)$ respectively, then $T_1^{-1}(a)\cdot T_2^{-1}(b)$ exists and is totally disconnected*(9).

Alternate definition of a simple plane web. Actually, we have shown that the following is an equivalent definition to that previously given.

Definition. A compact plane continuum W is a simple plane web provided there exist an upper semi-continuous collection G of mutually exclusive continua filling up W and another such collection H also filling up W such that (1) G is a dendron with respect to its elements and H is an arc with respect to its elements and (2) if g and h are elements of G and H respectively, then $g\cdot h$ exists and is totally disconnected. It has been shown [1] that condition (1) is unnecessary.

---

(9) G. T. Whyburn and some of his students have worked with monotone mappings onto dendrons. See [9] for references.

However, we cannot substitute for (1) in the above definition the condition (1′) that G is an arc with respect to its elements and so is H. Consider the plane continuum W obtained by subtracting from a circle C_0 plus its interior the interiors of five mutually exclusive circles C_1, C_2, C_3, C_4, C_5, each of which is tangent to C_0 internally. Now C_i is not both a subset of an element of G and also a subset of an element of H. Assume that C_j $(j=1, 2, 3)$ is not a subset of an element of G. Then there are uncountably many elements of G that intersect C_j. It may be seen that some element g_j of G separates a point P_j of C_j from $C_0+C_j+\cdots+C_5-C_j$ in W and intersects C_i $(i=0, 1, \cdots, 5)$ only if i is equal to j. No one of the elements g_1, g_2, g_3 separates the other two from each other in W. Hence, G is not an arc with respect to its elements.

Theorem 13. *If W is a simple plane web, there exist a dendron F in W, an arc α in F, an upper semi-continuous collection G of mutually exclusive continua filling up W and another such collection H also filling up W such that* (1) *G is a dendron with respect to its elements and H is an arc with respect to its elements,* (2) *if g and h are elements of G and H respectively, $g\cdot h$ exists and is totally disconnected,* (3) *each element of G intersects F in only one point, and* (4) *each element of H intersects α in only one point.*

Proof. Consider the dendron F and the upper semi-continuous collections G and H already defined. If α is the common part of F and the outer boundary of W, then F, α, G, and H satisfy the conditions of our theorem.

Theorem 14. *If W is a simple plane web, there exist a dendron F in W, an upper semi-continuous collection G of mutually exclusive continua filling up W and another collection H also filling up W such that* (1) *G is a dendron with respect to its elements and so is H,* (2) *if g and h are elements of G and H respectively, $g\cdot h$ exists and is totally disconnected,* (3) *each element of G intersects F in only one point and so does each element of H.*

Proof. The dendron F and the upper semi-continuous collection G are the same as those previously described.

We shall use the same symbols as those used in our definition of the arc H. Let F'' be the image of F' under T'. If h is the element of H'' containing V_4+V_6, let H_1'' denote the set of all continua h'' such that h'' is either V_1, h, or an element of H'' between V_1 and h. Using the methods of Theorem 12, it can be shown that there exists an upper semi-continuous collection H_2'' of mutually exclusive continua filling up $h+(\overline{R}-H_1''^*)$ such that (1) H_2'' is a dendron with respect to its elements, (2) h is a subset of an element of H_2'', (3) each element of H_2'' intersects F'' in only one point and V_6V_3 in a totally disconnected set, and (4) if g'' is an element of G'' intersecting an element h'' of H_2'', $g''\cdot h''$ is totally disconnected. We define H from $H_1''+H_2''$ in the same manner in which the arc H was defined from H''.

Theorem 14 may also be stated as follows.

THEOREM 14. *Suppose that W is a compact plane continuum and that T_1, T_2 are continuous monotone transformations of W such that if a and b are elements of $T_1(W)$ and $T_2(W)$ respectively, then $T_1^{-1}(a)$ and $T_1^{-2}(b)$ are nondegenerate sets whose common part exists and is totally disconnected. Then there exist monotone retractions R_1 and R_2 of W such that $R_1(W)$ is a dendron, $R_2(W)$ is the same dendron and if a and b are elements of this dendron, $R_1^{-1}(a)\cdot R_2^{-1}(b)$ exists and is totally disconnected.*

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THE UNIVERSITY OF TEXAS

ON GENERALIZED CONVEX FUNCTIONS

BY

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Reprinted from the
Transactions of the American Mathematical Society
Vol. 58, No. 2, pp. 220-230
September, 1945

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1. **Introduction.** Let $\{F(x)\}$ be a family of real functions defined in the interval $a<x<b$ and satisfying the following conditions:

(1) Each $F(x)$ is a continuous function of x in $a<x<b$.

(2) There is a unique member of the family which, at arbitrary x_1, x_2 satisfying $a<x_1<x_2<b$, takes on arbitrary values y_1, y_2 respectively.

By $F(x; x_1, y_1; x_2, y_2)$ we shall denote the member of $\{F(x)\}$ which takes on values y_1, y_2 at x_1, x_2, respectively. For a given function $f(x)$ we shall for brevity denote $F[x; x_i, f(x_i); x_j, f(x_j)]$ by $F_{i,j}(x)$. We shall further denote the interval $a<x<b$ by (a, b).

Definition. The real function $f(x)$, defined in (a, b), is a *sub-$F(x)$ function* provided the inequality

$$f\left(\frac{x_1+x_2}{2}\right) \leqq F_{1,2}\left(\frac{x_1+x_2}{2}\right) \tag{3}$$

holds for all x_1, x_2 satisfying $a<x_1<x_2<b$.

If the functions $F(x)$ are linear, then the class of sub-$F(x)$ functions is the class of convex functions. Since not all convex functions are continuous [4](1), it follows that (1), (2) and (3) do not imply that $f(x)$ is continuous. For a discussion of convex functions, see [2, 5, 6, 8].

We note that the above definition of a sub-$F(x)$ function is more general than that previously given [1], in which (3) was replaced by

$$f(x) \leqq F_{1,2}(x) \tag{4}$$

for all x_1, x_2, x satisfying $a<x_1<x<x_2<b$. Functions $f(x)$ satisfying (4) necessarily are continuous (Corollary 4, below); and for continuous functions $f(x)$, conditions (3) and (4) are equivalent (Lemma 3, below).

2. **The family $\{F(x)\}$.** We shall have need for the first two results given in this section. The third is included to show that for some families $\{F(x)\}$ satisfying (1) and (2) the class of sub-$F(x)$ functions is not topologically equivalent to the class of convex functions.

Lemma 1 [1]. *For a given x_0 with $a<x_0<b$, let $F_r(x)$ and $F_s(x)$ be two members of a family $\{F(x)\}$ satisfying* (1) *and* (2), *such that*

$$F_r(x_0) = F_s(x_0), \tag{5}$$

Presented to the Society, April 29, 1944; received by the editors June 12, 1944.

(1) Numbers in brackets refer to the references cited at the end of the paper.

$$(6) \qquad F_r(x) \not\equiv F_s(x) \qquad (a < x < b);$$

then $F_r(x) < F_s(x)$ *for all* x *in* (a, b) *on one side of* x_0, *while* $F_r(x) > F_s(x)$ *for all* x *in* (a, b) *on the other side of* x_0.

Proof. By (2), (5) and (6), $F_r(x) \neq F_s(x)$ in (a, b), except at x_0; consequently, by (1), on either side of x_0, one of $F_r(x)$, $F_s(x)$ is greater than the other. Suppose it could be the same one, say $F_s(x)$, which is greater on each side; we shall show that this is impossible.

Let x_1, x_2 satisfy $a < x_1 < x_0 < x_2 < b$, and consider the member $F_t(x)$ of $\{F(x)\}$, determined by $F_t(x) \equiv F[x;\ x_1,\ F_s(x_1);\ x_2,\ F_r(x_2)]$. We have $F_t(x_2) < F_s(x_2)$, so that $F_t(x) < F_s(x)$, $x_1 < x < b$; in particular,

$$(7) \qquad F_t(x_0) < F_s(x_0).$$

Similarly,

$$(8) \qquad F_t(x_0) > F_r(x_0).$$

Now (7) and (8) contradict (5).

LEMMA 2. *Let* x_0 *satisfy* $a < x_0 < b$ *and let* E *be a closed and bounded set of numbers in* (a, x_0) *(or in* (x_0, b)*); then for any real* y_0, M *there is a member* $F^*(x)$ *of* $\{F(x)\}$ *satisfying* $F^*(x_0) = y_0$ *and* $F^*(x) > M$ *for all* x *in* E.

Proof. For each x' in E there is an $h = h(x')$ such that if $I = I(x')$ denotes the interval $(x' - h, x' + h)$, we have

$$F(x;\ x_0,\ y_0;\ x',\ 2M) > M$$

for all x in I. By the Heine-Borel theorem there is a finite set $x_1', x_2', \cdots, x_n'$ of numbers in E such that each number in E is contained in at least one of $I(x_1'), I(x_2'), \cdots, I(x_n')$; therefore there is a finite set

$$R:\quad F_1(x), F_2(x), \cdots, F_n(x),$$

of members of $\{F(x)\}$ such that for each x' in E, we have

$$\max\,[F_1(x'), F_2(x'), \cdots, F_n(x')] > M.$$

By Lemma 1, some member of R satisfies the conditions specified for $F^*(x)$.

REMARK [7]. *Not all families* $\{F(x)\}$ *satisfying* (1) *and* (2) *are topologically equivalent to the family* $\{L(x)\}$ *of all nonvertical line segments terminating on* $x = a$, $x = b$.

Proof. Let the family $\{F(x)\}_0$ be determined for $-\infty < x < +\infty$ as follows: A function $F(x)$ is a member of $\{F(x)\}_0$ if and only if its graph in Cartesian coordinates is (i) a line of nonpositive slope or (ii) a continuous broken line, with break on the x-axis, such that the slope is positive for $y < 0$ and half as much for $y > 0$.

The family $\{F(x)\}_0$ satisfies conditions (1) and (2) in any interval (a, b).

Let ABC and $A'B'C'$ be triangles in the half-strip $a<x<b$, $y<0$, such that the lines AA', BB', CC' intersect in a point P of $a<x<b$, $y>0$; the slopes of AA' and BB' are negative while that of CC' is positive; and (see Desargues' Theorem [3]) the corresponding sides of the triangles intersect in collinear points in $a<x<b$, $y<0$.

Let $F_A(x)$, $F_B(x)$, $F_C(x)$ be the members of $\{F(x)\}_0$ whose graphs contain A, A'; B, B'; C, C', respectively. Then P is on $F_A(x)$ and on $F_B(x)$, but not on $F_C(x)$.

Suppose there were a reversible continuous transformation of the strip $a<x<b$ into itself carrying $\{F(x)\}_0$ into $\{L(x)\}$. Since a reversible continuous transformation preserves concurrency of graphs, triangles ABC and $A'B'C'$ are carried into triangles whose corresponding sides intersect in collinear points, so that, by the converse of Desargues' Theorem, the lines joining corresponding vertices are parallel or concurrent. This contradicts the fact that P is on $F_A(x)$ and on $F_B(x)$ but not on $F_C(x)$.

Actually we have shown more, that no plane domain D containing a segment of the x-axis can be mapped topologically on a plane domain D' in such a way that the graphs of the functions $\{F(x)\}_0$ in D are mapped on unbroken lines or line-segments in D'.

3. **Continuous sub-$F(x)$ functions.** We shall use the following lemma.

LEMMA 3. *If $f(x)$ is continuous in (a, b), then a necessary and sufficient condition that $f(x)$ be a sub-$F(x)$ function in (a, b) is that we have*

$$f(x) \leqq F_{1,2}(x) \tag{9}$$

for all x_1, x_2, x satisfying $a<x_1<x<x_2<b$.

Necessity. If (9) does not hold, then there exist x_3, x_4, x_5, with $a<x_3<x_5<x_4<b$, such that

$$f(x_5) > F_{3,4}(x_5). \tag{10}$$

By (10) and the continuity of $f(x)$ and of $F_{3,4}(x)$, there exist x_1, x_2, with $x_3 \leqq x_1 < x_5 < x_2 \leqq x_4$, such that

$$f(x) \geqq F_{3,4}(x) \qquad (x_1 \leqq x \leqq x_2), \tag{11}$$

the sign of equality holding at the end points but not elsewhere; then $F_{3,4}(x) \equiv F_{1,2}(x)$, so that (11) can be written as

$$f(x) > F_{1,2}(x) \qquad (x_1 < x < x_2). \tag{12}$$

Now (12) contradicts (3); hence if (9) does not hold, then $f(x)$ is not a sub-$F(x)$ function.

Sufficiency. If (9) holds, then (3) holds as a special case of (9), and consequently $f(x)$ is a sub-$F(x)$ function.

Our argument gives also the following result.

COROLLARY 1. *A necessary and sufficient condition that a function $f(x)$ continuous in (a, b) be a sub-$F(x)$ function is that for each pair of numbers x_1, x_2 satisfying $a<x_1<x_2<b$, there exist an x_3 satisfying $x_1<x_3<x_2$ such that $f(x_3) \leqq F_{1,2}(x_3)$.*

LEMMA 4. *A necessary and sufficient condition that a function $f(x)$ continuous in (a, b) be a sub-$F(x)$ function is that there exist a set E of numbers dense in (a, b) such that for each x_1, x_2 in E we have*

$$f\left(\frac{x_1 + x_2}{2}\right) \leqq F_{1,2}\left(\frac{x_1 + x_2}{2}\right). \tag{13}$$

Necessity. If $f(x)$ is a sub-$F(x)$ function, then the existence of a set E of numbers dense in (a, b) such that (13) holds for each x_1, x_2 in E follows from the definition of a sub-$F(x)$ function.

Sufficiency. As shown in Lemma 3, (12), if $f(x)$ is not a sub-$F(x)$ function, then there exist x_1, x_2 such that $f(x) > F_{1,2}(x)$ $(x_1<x<x_2)$.

Let x_0 satisfy $a<x_0<x_1$ and let x_3 and x_4 divide (x_1, x_2) into three equal parts, $x_3<x_4$. For each x' such that $x_3 \leqq x' \leqq x_4$ there is a y' and an $h=h(x')$ such that if $I=I(x')$ is the interval $(x'-h, x'+h)$, then

$$f(x) > F[x;\, x_0, F_{1,2}(x_0);\, x', y'] > F_{1,2}(x)$$

for all x in I. A finite number of the intervals I covers (x_3, x_4). By Lemma 1, one of the corresponding functions $F[x;\, x_0, F_{1,2}(x_0);\, x', y']$, which we shall denote by $F^*(x)$, accordingly satisfies the inequalities

$$f(x) > F^*(x) > F_{1,2}(x) \qquad (x_3 \leqq x \leqq x_4). \tag{14}$$

For any set E of numbers dense in (a, b), there are numbers x_5, x_6 in E such that $x_1<x_5<x_3<x_4<x_6<x_2$, for which

$$f(x_5) < F^*(x_5), \qquad f(x_6) < F^*(x_6). \tag{15}$$

Since we have $x_3<(x_5+x_6)/2<x_4$, we find from (14), (15) and Lemma 1 that

$$f\left(\frac{x_5 + x_6}{2}\right) > F^*\left(\frac{x_5 + x_6}{2}\right) > F_{5,6}\left(\frac{x_5 + x_6}{2}\right).$$

Therefore if $f(x)$ is not a sub-$F(x)$ function, there does not exist a set E of numbers dense in (a, b) for which (13) is satisfied.

Actually our method of proof gives more, as stated in the following corollary.

COROLLARY 2. *A necessary and sufficient condition that a function $f(x)$ continuous in (a, b) be a sub-$F(x)$ function is that there exist a positive number ϵ*

and a set E of numbers dense in (a, b) such that for each pair of numbers x_1, x_2 in E satisfying $a<x_1<x_2<b$, there is an x_3 in E such that

$$x_1+\epsilon(x_2-x_1)<x_3<x_2-\epsilon(x_2-x_1),$$

for which $f(x_3) \leqq F_{1,2}(x_3)$.

As stated in Corollary 1, if the function $f(x)$ is continuous in (a, b) and if for each pair of numbers x_1, x_2 satisfying $a<x_1<x_2<b$ there is an x_3 satisfying $x_1<x_3<x_2$, such that $f(x_3) \leqq F_{1,2}(x_3)$, then $f(x)$ necessarily is a sub-$F(x)$ function. However, there does exist a function $f(x)$ continuous in (a, b), and a set E dense in (a, b), such that for each pair of numbers x_1, x_2 in E satisfying $a<x_1<x_2<b$, there is an x_3 in E satisfying $x_1<x_3<x_2$, such that $f(x_3) \leqq F_{1,2}(x_3)$, and such that $f(x)$ is not a sub-$F(x)$ function. For let $(a, b)=(0, 1)$, and define $f(x)$ as follows:

$$\begin{aligned} f(x) &= 1/2 && (1/3 \leqq x \leqq 2/3), \\ f(x) &= 1/4 && (1/9 \leqq x \leqq 2/9), \\ f(x) &= 3/4 && (7/9 \leqq x \leqq 8/9), \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot && \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ f(x) &\text{ continuous in } (0, 1). \end{aligned}$$

If E is the set of all numbers x satisfying

$$1/3<x<2/3, \text{ or } 1/9<x<2/9, \text{ or } 7/9<x<8/9, \text{ or } \cdots,$$

and $\{F(x)\}$ is the set of all nonvertical straight line segments in the strip $0<x<1$, then $f(x)$ satisfies the conditions set forth above.

THEOREM 1. *If $f(x)$ is a sub-$F(x)$ function in (a, b), then for any fixed x_1, x_2 satisfying $a<x_1<x_2<b$, and for all ρ in (a, b) dividing (x_1, x_2) rationally (internally or externally), the points $[\rho, f(\rho)]$ lie on the graph of a uniquely determined continuous sub-$F(x)$ function.*

Proof. For fixed x_1, x_2 with $a<x_1<x_2<b$, let $F_s(x)$ be a member of $\{F(x)\}$ satisfying $F_s(x_1) \geqq f(x_1)$, $F_s(x_2) \geqq f(x_2)$; then for all non-negative integers m, n such that $m/2^n \leqq 1$, we have

$$f\left[x_1+\frac{m}{2^n}(x_2-x_1)\right] \leqq F_s\left[x_1+\frac{m}{2^n}(x_2-x_1)\right]. \tag{16}$$

For, by Lemma 1, we have

$$F_{1,2}(x) \leqq F_s(x) \qquad (x_1 \leqq x \leqq x_2). \tag{17}$$

From (3) and (17) it follows that we have

$$f\left(x_1 + \frac{x_2 - x_1}{2}\right) \leqq F_{1,2}\left(x_1 + \frac{x_2 - x_1}{2}\right) \leqq F_s\left(x_1 + \frac{x_2 - x_1}{2}\right).$$

The inequality (16) follows by induction.

Next, we shall show that, for the above x_1, x_2, if $F_r(x)$ is a member of $\{F(x)\}$ satisfying

$$F_r(x_1) \geqq f(x_1), \qquad F_r(x_2) \leqq f(x_2),$$

then for all positive integers m, n such that

$$\frac{m}{2^n} \geqq 1, \qquad x_1 + \frac{m}{2^n}(x_2 - x_1) < b,$$

we have

$$f\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right] \geqq F_r\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right]. \tag{18}$$

From Lemma 1 we obtain

$$F_r(x) \leqq F_{1,2}(x) \qquad (x_2 \leqq x < b). \tag{19}$$

For any positive integers m, n such that $1 \leqq m/2^n \leqq 2$, there is a non-negative integer m' such that $(m+m')/2^{n+1} = 1$. Since x_2 is the average of the numbers

$$x_1 + \frac{m'}{2^n}(x_2 - x_1), \qquad x_1 + \frac{m}{2^n}(x_2 - x_1),$$

and since from (16) we have

$$f\left[x_1 + \frac{m'}{2^n}(x_2 - x_1)\right] \leqq F_{1,2}\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right],$$

it follows from (3), (19) and Lemma 1 that if $x_1 + m(x_2 - x_1)/2^n$ is in (a, b), then

$$f\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right] \geqq F_{1,2}\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right] \geqq F_r\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right].$$

By induction we have (18) also for $m/2^n > 2$, provided only that $x_1 + m(x_2 - x_1)/2^n$ is in (a, b).

Likewise, we find that if $F_t(x)$ is a member of $\{F(x)\}$ satisfying

$$F_t(x_1) \leqq f(x_1), \qquad F_t(x_2) \geqq f(x_2),$$

then for all non-negative integers m, n such that

$$x_1 - \frac{m}{2^n}(x_2 - x_1) > a,$$

we have

$$f\left[x_1 - \frac{m}{2^n}(x_2 - x_1)\right] \geqq F_t\left[x_1 - \frac{m}{2^n}(x_2 - x_1)\right]. \tag{20}$$

Let j be a positive integer sufficiently large so that the number

$$x_0 = x_1 - \frac{1}{2^j}(x_2 - x_1)$$

satisfies $a<x_0<x_1$. By (16) and (18), for all non-negative integers m, n such that $m/2^n \leqq 1$, we have

$$\begin{aligned} F_{1,2}\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right] &\geqq f\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right] \\ &\geqq F_{0,1}\left[x_1 + \frac{m}{2^n}(x_2 - x_1)\right]. \end{aligned} \tag{21}$$

For numbers $x = x_1 + (m/2^n)(x_2 - x_1)$, where m, n are non-negative integers such that $m/2^n \leqq 1$, designate by W the set of points $[x, f(x)]$. The boundedness of W follows from (1) and (21). Since the set of abscissas of points of W is dense in (x_1, x_2), either the closure $\overline{W}$ of W is a continuous curve which is the graph of a single-valued function $y = \overline{W}(x)$ continuous in (x_1, x_2), or else there exists an x_3 satisfying $x_1 < x_3 < x_2$, and numbers y_1, y_2 with $y_1 < y_2$, such that the points with coordinates (x_3, y_1) and (x_3, y_2) belong to $\overline{W}$.

Assume that there do exist such x_3, y_1, y_2. Consider members $F^*(x)$ and $F^{**}(x)$ of $\{F(x)\}$ determined by

$$F^*(x_1) = f(x_1), \qquad F^*(x_3) = F^{**}(x_3) = \frac{y_1 + y_2}{2}, \qquad F^{**}(x_2) = f(x_2).$$

Since (x_3, y_2) is a point of $\overline{W}$, it follows from the continuity of $F^*(x)$ and of $F^{**}(x)$, and from (18) and Lemma 1, that there are positive integers j, k, with $j/2^k < 1$, such that the number $x_4 = x_1 + (j/2^k)(x_2 - x_1)$ is distinct from x_3 and satisfies

$$f(x_4) > F^*(x_4), \qquad f(x_4) > F^{**}(x_4). \tag{22}$$

If $x_4 > x_3$, let

$$x_5 = x_1 + \frac{j+1}{2^k}(x_2 - x_1).$$

From (20) we obtain $f(x) \geqq F_{4,5}(x)$ for all x less than x_4 satisfying

$$x = x_1 + \frac{m}{2^n}(x_2 - x_1),$$

where m, n are positive integers. Since the point (x_3, y_1) of $\overline{W}$ has ordinate $y_1 < F^{**}(x_3)$, it follows from the continuity of $F^{**}(x)$ and of $F_{4,5}(x)$ that for some number x_6 satisfying $x_3 < x_6 < x_4$, we have

$$F^{**}(x_6) = F_{4,5}(x_6). \tag{23}$$

But from (18) and the definition of $F^{**}(x)$, we obtain $F^{**}(x_2) \geqq F_{4,5}(x_2)$, so that there is a number $x_7 > x_4$ such that

$$F^{**}(x_7) = F_{4,5}(x_7). \tag{24}$$

Now (22), (23) and (24) contradict (2), whence the continuity of $\overline{W}(x)$ in the interval $x_1 < x < x_2$ is proved on the assumption that $x_4 > x_3$.

Considering $F^*(x)$ instead of $F^{**}(x)$, we can modify our argument so as to prove $\overline{W}(x)$ continuous in the interval $x_1 < x < x_2$ on the assumption that $x_4 < x_3$.

By considering suitable x_1', x_2', with $a < x_1' < x_1 < x_2 < x_2' < b$, we see that the range of $\overline{W}(x)$ can be extended so that $\overline{W}(x)$ remains continuous in the closed interval $x_1 \leqq x \leqq x_2$.

We shall show that for any positive rational number $p/q < 1$ we have

$$f(x) = \overline{W}(x) \qquad \left[x = x_1 + \frac{p}{q}(x_2 - x_1)\right]. \tag{25}$$

Assume that there is a rational number p_0/q_0 with $0 < p_0/q_0 < 1$, such that

$$f(x_0) \neq \overline{W}(x_0) \qquad \left[x_0 = x_1 + \frac{p_0}{q_0}(x_2 - x_1)\right]. \tag{26}$$

By the method of the last paragraph, there is a function $G(x)$ continuous in the interval $x_1 \leqq x \leqq x_0$ such that for all non-negative integers m, n with $m/2^n \leqq 1$ we have

$$f(x) = G(x) \qquad \left[x = x_1 + \frac{m}{2^n}(x_0 - x_1)\right]. \tag{27}$$

From the continuity of $\overline{W}(x)$ and of $G(x)$, and from (26) and (27), it follows that there is a positive h such that

$$G(x) \neq \overline{W}(x) \qquad (x_0 - h < x \leqq x_0). \tag{28}$$

There exist positive integers m, n, m', n' such that

$$\frac{m}{2^n} < 1, \qquad \frac{m}{2^n}\left(\frac{p_0}{q_0}\right) = \frac{m'}{2^{n'}}, \qquad \frac{m}{2^n}(x_0 - x_1) > x_0 - x_1 - h.$$

But for

$$x_8 = x_1 + \frac{m'}{2^{n'}}(x_2 - x_1) = x_1 + \frac{m}{2^n}(x_0 - x_1),$$

we have

$$x_0 - h < x_1 + \frac{m}{2^n}(x_0 - x_1) = x_8 < x_0 \tag{29}$$

and

$$G(x_8) = f(x_8) = \overline{W}(x_8). \tag{30}$$

Since (29) and (30) contradict (28), (25) follows.

Since $f(x)$ is a sub-$F(x)$ function and since $\overline{W}(x)$ is continuous in the interval $x_1<x<x_2$, it follows from (25) and Lemma 4 that $\overline{W}(x)$ is a sub-$F(x)$ function in (x_1, x_2).

By considering suitable x_1', x_2', with $a<x_1'<x_1<x_2<x_2'<b$, we see that the range of $\overline{W}(x)$ can be extended so that $\overline{W}(x)$ remains continuous in (a, b) and satisfies the conditions of Theorem 1.

4. **Bounded sub-$F(x)$ functions.** We shall use the theorem proved in this section in establishing a more general result in §5.

LEMMA 5. *If $f(x)$ is a sub-$F(x)$ function in the interval (a, b), and is bounded from above in an interval (x_1, x_2) with $a<x_1<x_2<b$, then:*

(i) *For any x_3 with $a<x_3<x_1$ there is an $F^*(x)$ such that $F^*(x_3)=f(x_3)$ and such that $F^*(x)\geqq f(x)$ for all x satisfying $x_3\leqq x\leqq x_2$.*

(ii) *For any x_4 with $x_2<x_4<b$ there is an $F^{**}(x)$ such that $F^{**}(x_4)=f(x_4)$ and such that $F^{**}(x)\geqq f(x)$ for all x satisfying $x_1\leqq x\leqq x_4$.*

Proof. By Lemma 2 there is an $F^*(x)$ satisfying $F^*(x_3)=f(x_3)$, such that $F^*(x)>f(x)$ for all x satisfying $x_1\leqq x\leqq x_2$. For any x' in (x_3, x_1) there is an x'' in (x_1, x_2) such that x' divides (x_3, x'') rationally. Therefore, by Theorem 1 and Lemmas 1 and 3, we have $F^*(x)\geqq f(x)$ for all x satisfying $x_3\leqq x\leqq x_2$.

In like manner, it can be shown that there is an $F^{**}(x)$ satisfying the second part of Lemma 5.

The following corollary follows from Lemma 5 and the continuity of the members of $\{F(x)\}$.

COROLLARY 3. *If $f(x)$ is a sub-$F(x)$ function in the interval (a, b), and is bounded from above in some subinterval of (a, b), then $f(x)$ is bounded from above in every closed subinterval of (a, b).*

THEOREM 2. *If $f(x)$ is a sub-$F(x)$ function in the interval (a, b), and is bounded from above in some subinterval of (a, b), then $f(x)$ is continuous in (a, b).*

Proof. It will be shown that $f(x)$ is continuous at an arbitrary point x_0 of (a, b).

If $I: x_1 \leqq x \leqq x_2$ is a closed subinterval of (a, x_0), then by Corollary 3, $f(x)$ has an upper bound in I. By Lemma 5, there is an $F^*(x)$ such that

$$F^*(x_0) = f(x_0), \qquad F^*(x) \geqq f(x) \qquad (x_1 \leqq x \leqq x_0). \tag{31}$$

For any x' in (x_0, b), there is an x'' in (x_1, x_0) such that x_0 divides (x'', x') rationally. By Theorem 1 and Lemma 3 we have

$$F^*(x) \leqq f(x \qquad (x_0 \leqq x < b). \tag{32}$$

Similarly, there is an x_3 in (x_0, b) and an $F^{**}(x)$ satisfying

$$F^{**}(x_0) = f(x_0), \qquad F^{**}(x) \geqq f(x) \qquad (x_0 \leqq x \leqq x_3), \tag{33}$$

and

$$F^{**}(x) \leqq f(x) \qquad (a < x \leqq x_0). \tag{34}$$

From (31)–(34) and the continuity of the members of $\{F(x)\}$, the continuity of $f(x)$ at x_0 follows. Since x_0 is arbitrary, $a < x_0 < b$, the continuity of $f(x)$ in (a, b) is established.

Corollary 4. *If $f(x)$ is defined in (a, b) and satisfies $f(x) \leqq F_{1.2}(x)$ for all x_1, x_2, x with $a < x_1 < x < x_2 < b$, then $f(x)$ is continuous in (a, b).*

There is a direct proof [1] of Corollary 4 which is simpler than the above proof of Theorem 2.

5. **Measurable sub-$F(x)$ functions.** We shall use Theorem 2 and the following lemma to obtain a result which is more general than Theorem 2.

Lemma 6. *If $f(x)$ is a sub-$F(x)$ function in (a, b) and is bounded from above on a set E of positive measure in (a, b), then $f(x)$ is bounded from above in some subinterval of (a, b).*

Proof. There is a closed subinterval $x_1 \leqq x \leqq x_2$ of (a, b) such that the measure of the part of E in (x_1, x_2) is greater than $2(x_2 - x_1)/3$. Let x_3 and x_4, with $x_3 < x_4$, divide (x_1, x_2) into three equal parts, and let x' be any value satisfying $x_3 \leqq x' \leqq x_4$. Since the measure of the part of E in (x_1, x_2) is greater than $2(x_2 - x_1)/3$, it follows that the measure of the part of E in

$$[x' - (x_4 - x_3),\ x' + (x_4 - x_3)]$$

is greater than $x_4 - x_3$; consequently, there is a $\delta(x')$ less than $x_4 - x_3$ such that $x' - \delta(x')$ and $x' + \delta(x')$ both belong to E.

By assumption there is an M such that we have $f(x) < M$ for all x in E. By Lemma 2, there is an $F^*(x)$ such that $F^*(x) > M$ for all x satisfying $x_1 \leqq x \leqq x_2$. From Lemma 1 and the fact that $f(x)$ is a sub-$F(x)$ function, it follows that

$$F^*(x) > F[x;\ x - \delta(x),\ f(x - \delta(x));\ x + \delta(x),\ f(x + \delta(x))] \geqq f(x)$$
$$(x_3 \leqq x \leqq x_4).$$

Since $F^*(x)$ is continuous, we conclude that $f(x)$ is bounded, $x_3 \leqq x \leqq x_4$.

THEOREM 3. *If $f(x)$ is a sub-$F(x)$ function in (a, b), and is bounded from above on a set E of positive interior measure in (a, b), then $f(x)$ is continuous in (a, b).*

Proof. Let E_0 be a closed subset of E having positive measure. By Lemma 6 there is a subinterval of (a, b) on which $f(x)$ is bounded from above, so that the continuity of $f(x)$ in (a, b) follows from Theorem 2.

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